LEŚNIEWSKI'S

COMPUTATIVE PROTOTHETIC

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Contents

Abstract

Abstract of thesis submitted by A. O. V. Le Blanc for the Degree of Doctor of Philosphy, and entitled *Leśniewski's Computative Protothetic*. Submitted January, 1991.

The logician Stanisław Leśniewski devoted most of his academic life to the development of a system of foundations of mathematics, which consists of three deductive theories: protothetic, ontology, and mereology. Protothetic is the most general of these theories, logically prior to the others; it has been described by its creator as a unique extension of the classical 'theory of deduction' or 'propositional calculus', though this theory differs from more usual versions in many respects. The 'standard' system of protothetic is developed by a rule of procedure corresponding to the traditional style of development incorporating substitution and detachment, but including directives for definition and extensionality.

Leśniewski also developed systems of protothetic whose rule of procedure does not contain directives for substitution or detachment, and whose style of development has been described as 'computative' or as involving 'automatic verification'. The directives may be said to resemble Peirce's zero/one verification method, though they are extended to allow verification and rejection of expressions containing variables in all semantic categories, and having various numbers of possible 'values'. Only an informal summary of Leśniewski's work on these systems survives.

This thesis examines computative protothetic historically, informally, and formally. It contains a set of directives for a system of computative protothetic which is as close as possible to the lost directives of Leśniewski's own systems.

Declaration

No portion of this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Education and Research

A. B.	1973	Spring Hill College, Mobile, Alabama	Greek, Latin, Philosophy
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Areas of research include Stanisław Leśniewski's work and systems, axiomatisations of the theory of deduction, certain areas in non-standard logic, computer assisted proofs based on substitution and detachment, and computer generated matrices for independence proofs.

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This thesis was typeset and printed using TEX and several related utilities, using additional special characters which I designed to imitate those used in the original works of Leśniewski and in the best printed works about his systems. I am grateful to Prof. Donald E. Knuth, who designed TEX and made it widely available, to the hundreds of others who helped to improve it and to implement it so widely, and to the many who helped me learn to use it. I must also thank the Computing Centre of the University of Manchester for giving me the facilities for typesetting and for printing this work.

It has been difficult to complete the thesis, which has required long hours of tedious detailed research, while being employed full time in an unrelated field. I am grateful to Dr. John R. Chidgey for his help in proofreading, and to Prof. Czesław Lejewski for his frequent encouragement and for helping me try to conform to ever higher standards of clarity and of precision.

1. Introduction

This work aims to describe and present the systems of *computative protothetic* as precisely as possible. The original systems of computative protothetic have been lost. Only a brief sketch of them survives; it is incomplete and, in many respects, ambiguous. We shall resolve these ambiguities in a manner which is as close as possible to the spirit of the original systems of computative protothetic.

Leśniewski believed that many methods should be used to present a deductive theory as clearly as possible. We shall employ the following methods, all of which Leśniewski himself used in various publications and in his lectures: historical summary, informal description, comparison with other deductive theories, comparison between equivalent systems, informal and formal presentation of the system, and proofs of metatheorems about systems. The combined effect of these tactics should be a sharp focus on systems of computative protothetic, making them more accessible to logicians and philosophers who are unfamiliar with Leśniewski's work.

This introduction presents general information which we need if we are to understand what protothetic is.

1.1. Leśniewski's deductive theories

The logician Stanisław Leśniewski (1886–1939) devoted the latter decades of his life to the development of a system of foundations of mathematics[1]. The system consists of three deductive theories: *protothetic*, *ontology*, and *mereology*. Leśniewski claimed that the combination of these theories formed 'one of the possible foundations of the whole system of mathematical disciplines'[2].

The theory of parts and of collections or wholes actually consisting of their parts is named 'mereology', which means the 'science of parts'[3]. A 'collection' in this sense is unlike, for example, a 'set' in contemporary mathematics or a 'species' in medieval philosophy. One can easily define in mereology the terms 'in' or 'on' and 'point'[4], so that this theory can serve as the basis for systems of geometry[5]. In 1916 Leśniewski claimed that mereology's term 'manifold' [*mnogość*] fulfilled the essential conditions which Cantor wished to hold of a '*Menge*'[6]. Mathematicians have developed from the ideas of Cantor and others a quite different theory, now known as 'set theory'. Many terms appear in 'set theory' which Leśniewski used in completely different senses in his lectures and publications. Sometime after

[1] Bird75 and Lejewski67 contain general information about Leśniewski and his work. The Bibliography contains full references for all cited works.

[2] Leśniewski27, p. 165.

[3] The fundamental sources for mereology are Leśniewski28a, Leśniewski29a, Leśniewski30a, and Leśniewski31a. Sobociński55 contains an introduction to mereology in English. For the shortest known axiom systems and for a list of the major works on mereology see LeBlanc83.

[4] The term 'point' is used here in the geometrical sense. Leśniewski preferred to speak of a point-moment.

[5] Sobociński49, p. 12, and Luschei62, p. 150.

[6] Leśniewski16, p. 5.

1923 but before 1927 he began to abandon this terminology in order to avoid unnecessary confusion[7]. At that time he coined the word 'mereology'.

Many terms appear in mereology which cannot be defined by means of the primitive terms of the system. Some of them can be defined with the help of the term 'is'. Leśniewski saw these terms as part of a theory more general than mereology and logically prior to it. This theory of objects, within which existence can be discussed, is based in its standard formulation on the primitive term 'is', so it seems appropriate to call it 'ontology', which means the 'science of being'. In 1920 Leśniewski constructed the first axiom system for ontology[8]. He describes it as a modernised form of 'traditional logic' whose content resembles that of Schröder's 'Calculus of Classes', if one regards this as including the theory of 'individuals'[9]. The fundamental terms and operations of the theory of numbers can be defined in ontology[10], so that it can serve as the basis for arithmetic.

Many terms appear in ontology which cannot be defined by means of the term 'is'. They are part of a theory more general than ontology and logically prior to it, the theory which Leśniewski called 'protothetic'[11], which means 'concerned with first or basic theses'. In 1922 Leśniewski constructed the first system of protothetic, in which all these terms are defined[12]. He describes this theory as the most fundamental logical and mathematical theory[13], a unique extension of the classical 'theory of deduction' or 'propositional calculus'[14]. The nature of the extensions incorporated into protothetic will be explained later.

Leśniewski and his followers have investigated the foundations of his deductive theories extensively. They have constructed many mutually equivalent systems of protothetic, ontology, and mereology[15]. These systems are based on various axioms, often containing different primitive terms. Some systems, in particular systems of ontology, have *directives* (rules of procedure) which differ from the 'standard' directives of protothetic and ontology[16]. Changes in the directives alter the 'deductive structures' of the systems to which they apply.

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[7] Leśniewski38, p. 57. Although he had coined the word 'mereology' by 1927, he was still referring to it as the 'theory of classes' in the following year; cf. Leśniewski28.

[8] Leśniewski31a, pp. 158–9.

[9] Leśniewski27, p. 166. The origins of ontology are discussed in Leśniewski31a, pp. 153–70. Introductions to ontology in English can be found in Lejewski58 and in Henry72. The fundamental theorems of ontology are proved in Sobociński34. The directives of ontology can be found in Leśniewski30.

[10] Cf. Luschei62, p. 148, and the references given there.

[11] This term had been coined by 1927. Previously Leśniewski called the theory 'logistic', but does not seem to have used this term (except when referring to his earlier usage) after 1928; cf. Leśniewski28.

[12] Leśniewski29, p. 36. The fundamental sources for protothetic are Leśniewski29, Leśniewski38, Leśniewski39, Sobociński60 with its continuations, and Słupecki53.

[13] Leśniewski29, p. 14.

[14] Leśniewski38, pp. 4–5.

[15] Examples of systems of protothetic can be found in Sobociński60, Sobociński61, Sobociński61a, and LeBlanc85. Examples of systems of ontology can be found in Lejewski58 and in Lejewski77. Examples of systems of mereology can be found in Leśniewski30a, Leśniewski31a, Lejewski55, and LeBlanc83.

[16] The directives of the standard system of protothetic $\mathfrak{S}_5$ are specified in Leśniewski29, pp. 59–78. Nonstandard systems of protothetic are discussed in Leśniewski29, pp. 35–50. The directives of the standard system of ontology appear in Leśniewski30. Nonstandard systems of ontology appear in Lejewski58 and Lejewski77.

Hence these systems provide new perspectives from which the underlying theories can be studied.

In their 'official' forms, Leśniewski's deductive systems employ a traditional substitution and detachment style of development. However all deductions in ontology and mereology published by Leśniewski and his followers use a variety of 'natural deduction'. Leśniewski believed that this informal style of reasoning conformed more closely to his logical intuitions, and that the more formal systems in fact codified these intuitions[17]. In other words, he regarded his informal proofs as outlines of formal proofs in the 'official' systems, but historically the axioms and directives of his 'official' systems grew from his informal proofs and his philosophical reflections. He constructed formal systems of both mereology and ontology long before he formalised these theories; that is, he wrote axioms and proved theorems from them using his style of 'natural deduction'[18].

1.2. Formalised deductive systems

Leśniewski was most careful to distinguish between *formal* and *informal* language. Formal language is characterised by the use of technical vocabulary and by extreme care in the use of words. Informal language includes everyday words and expressions. The drawing of comparisons occurs only in informal language. Quotation marks occur only in informal language, and they are used in at least two ways: to form common names for words and expressions to which reference is made, and to indicate terms and expressions which are not used in accordance with Leśniewski's formal terminology. In cases of particular danger, Leśniewski emphasises that something is not stated in formal language by using warning phrases such as these: 'sketch', 'outline', 'a general characterisation', 'I have convinced myself', 'with no pretence to exactness', 'so-called', 'freely speaking', etc. We shall attempt to exercise the same care.

A *system* consists of a series of sentences called *theses*. Theses are not abstractions or 'propositions' but material objects produced by human activity in a particular place. The theses of a system must be finite in number, but this number usually increases in the course of time as we add new theses to the system. A philosophical book might be an example of a formal system. In ancient Greek the term 'system' can mean a collection of objects. Leśniewski may have been influenced by Dedekind, who, according to Frege and Leśniewski[19], used it in the sense of a collection of objects in Dedekind88. David Hilbert may have invented and certainly popularised the phrase 'axiom system'[20].

A *deductive system* is a formal system which begins with axioms and which grows by adding to the system theses which are in some way legitimate additions. Most legitimate additions might be described as inferences or deductions from earlier theses, but in many cases this terminology seems inappropriate. For example, we may add definitions to a deductive system, but it is not reasonable to say that a definition is an inference. Euclid's *Elements* is an example of a deductive system.

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[17] Cf. Leśniewski29, p. 78.

[18] Examples are found in Leśniewski16 and in Leśniewski27 and its continuations.

[19] Leśniewski27, pp. 191–2.

[20] The earliest example of the phrase 'axiom system' currently known to me occurs in a letter of Hilbert's in 1899; see Frege76, p. 65.

A *deductive theory* is an abstraction: we say that a number of mutually equivalent deductive systems express or embody the same theory. Thus Euclidean geometry is a deductive theory embodied in all deductive systems which are equivalent in content to the system of Euclid's *Elements*.

A *formalised* system is a deductive system with *directives* or rules of procedure. These should determine unambiguously whether or not it is legitimate to add a given expression to the system as a new thesis. Frege's *Grundgesetze*[21] contained the first formalised deductive system. Leśniewski remarked that the deductive system of the *Grundgesetze* is superior to those created by later logicians because it was so carefully formalised that one can *prove* that the system is inconsistent[22]. In the deductive systems of Chwistek and of von Neumann, he showed how to introduce contradictions while observing all of the restrictions stated explicitly by the authors in their directives[23].

The formalisation of a deductive system requires the formal statement of its directives. Directives which are stated informally are very likely to be unclear. Clear directives require special technical terms which must be carefully defined. For example, one directive of the 'standard' system of protothetic $\mathfrak{S}_5$ states that if A is the last thesis belonging to the system of protothetic, you may add an expression B to the system as a new thesis immediately after A if for some C — '$B \ \varepsilon$ cnsqsbstp (A, C)'[24]. We can interpret this last expression as 'B is derivable from C by means of a correct substitution in protothetic with respect to A'[25]. The term 'cnsqsbstp' is defined in a series of 'terminological explanations' with the help of other terms, such as those which can be interpreted as 'a variable bound by B in C', 'a function', and 'a term in C which is suited to be a constant of protothetic relative to B'. The definition of each such term must tell us how to determine whether or not the term applies to a given expression by performing a 'combinatorial' decision procedure. We must be able to complete this procedure in a finite number of steps, and without needing to examine any expressions except those belonging to some given finite domain[26].

Leśniewski's method of formulating directives is one of his greatest contributions to logic. A deductive system constructed according to his methodology is particularly well suited to formal metalogical investigations.

1.3. Semantic categories

The characteristic concept of *semantic categories* in Leśniewski's systems corresponds in some respects to the 'theory of types' in the system of Whitehead and Russell[27]. Discoveries early in this century convinced most logicians of the need for something like Whitehead

[21] Frege93 and Frege03. One might argue that Frege's earlier work *Begriffsschrift* contained the first formalised deductive system, but its directives are not specified with the same care as those in the *Grundgesetze*. Cf. Frege79.

[22] I learned of this unpublished remark from Prof. Czesław Lejewski. Cf. Leśniewski27, pp. 166 and 168, and Leśniewski29, pp. 78–81.

[23] Leśniewski29, p. 79.

[24] Leśniewski29, p. 76.

[25] Cf. Leśniewski29, p. 73.

[26] Cf. Leśniewski31, p. 301.

[27] WhiteheadRussell10, pp. 37–65.

and Russell's theory of logical types. In 1921 Leśniewski constructed his own 'theory of types', which he later described as a simpler but more general version of Whitehead and Russell's theory[28]. He said little about this theory beyond mentioning that it used different shapes of parentheses[29] and commenting that

> Even at the moment when I constructed it, I considered my 'theory of types' as merely an insufficient mitigant [*Palliativ*] which, without threatening me with the 'antinomies', would at least temporarily enable me ... to use all the kinds of function variables which I wanted to use[30].

In 1922 he replaced this theory with his concept of 'semantic categories', which resembled the theory of types in its formal consequences, but which had a completely different philosophical basis[31]. The formal similarity between Leśniewski's 'theory of types' and his 'concept of semantic categories' must have been very close, and Professor Lejewski seems to give the best explanation of the difference between them:

> It is more likely than not that the notion of <a> logical type as a kind of extra-linguistic entity appeared to Leśniewski to be highly suspicious, and his logical and philosophical conscience ceased to worry him only when he saw that instead of postulating hierarchies of logical types he could talk about hierarchies of linguistic expressions[32].

Leśniewski presented his concept of semantic categories only as applied to the directives of protothetic and ontology. He never published a complete philosophical discussion of the subject, though his followers have often discussed it in relation to natural languages[33].

In very informal terms, we may say that Leśniewski classifies some expressions depending on the way in which they have meaning. A *sentence* has meaning by being true or false, and in this respect all sentences belong to the same semantic category. In formal languages there exist sentence-like expressions which are neither true nor false, but which must be regarded as belonging to the same category as sentences. For example, a propositional variable is not a sentence, because it is neither true nor false, but it belongs to the same category as sentences.

Another category recognised by Leśniewski is the category of *names*, which have meaning by attempting to refer to objects. He makes no distinction of *category* between common and proper names; in this respect Leśniewski abandons the tradition of Frege and returns to the approach of late classical and mediæval Aristotelian logicians. Once again, variables and other expressions which are used like names belong to the same semantic category as names, even though they clearly do not name anything. If we compare the two sentences

If Fido is a dog, then Fido has fleas.
For all A — if A is a dog, then A has fleas.

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[28] Leśniewski29, p. 13.

[29] Leśniewski29, p. 44.

[30] Ibid.

[31] Leśniewski29, p. 14.

[32] Lejewski65, p. 190.

[33] See Ajdukiewicz35, Lejewski65, and Lejewski79. Lejewski65 contains a particularly good informal introduction to semantic categories.

we can see that the variables 'A' function in much the same way as the corresponding names 'Fido', even though we cannot claim that the two variables name any objects.

When we recognise the categories of sentences and of names, we see that in some sense the expressions in them 'have meaning' in quite different ways. After reflecting on the difference, Leśniewski decided that meaningful terms or expressions which accept arguments of different semantic categories must themselves belong to different semantic categories. For example, suppose we have two sentences in a formal language:

Φ(Fido)
Φ(it is raining)

Leśniewski was unable to conceive of the two 'Φ's having the same meaning when one accepts a name argument and the other accepts a sentence argument.

In general in Leśniewski's systems there are two and only two basic categories: sentences and names. (No terms or expressions in systems of protothetic belong to the category of names or to any category derived from that category.) Other categories are introduced as categories of *functors*, which are expressions accepting a specified number of arguments to form a complete function. This complete function belongs to some specified semantic category. Each argument of the function must belong to some specified category. An expression in the category of sentences may appear as an entire thesis, as the contents of the scope of a quantifier, or as an argument of a function. An expression in the category of names may appear only as an argument of a function. Any category in the system except that of sentences and that of names may contain expressions which are the functors of functions and expressions which are the arguments of functions.

The concept of semantic categories led Kazimierz Ajdukiewicz to develop his *index notation*, which is convenient for naming and referring to semantic categories[34]. The following is a recursive 'definition' of a legitimate index:

(1) The letter 's' is a legitimate index.

(2) The letter 'n' is a legitimate index.

(3) A 'fraction' is a legitimate index if it has one 'numerator' which is a legitimate index, and one or more 'denominators' each of which is a legitimate index.

The index 's' represents the semantic category of sentences. The index 'n' represents the semantic category of names. A 'fraction' represents a functor which, when completed with arguments in the respective semantic categories represented by its 'denominators', forms a function in the semantic category represented by its 'numerator'. Thus, for example, the functor of propositional negation belongs to the category with the index '$\frac{\mathrm{s}}{\mathrm{s}}$'. The propositional functors of implication, alternation, conjunction, and equivalence belong to the category with the index '$\frac{\mathrm{s}}{\mathrm{s\ s}}$'. In the English sentence 'This is a green house', the word 'is' has the index '$\frac{\mathrm{s}}{\mathrm{n\ n}}$', and the word 'green' has the index '$\frac{\mathrm{n}}{\mathrm{n}}$', at least if we analyse the sentence in accordance with traditional grammar.

It is impossible to have a functor which belongs to the same semantic category as one of its arguments. If such a function existed, its index would need to be the same as

[34] This notation was first introduced and used in Ajdukiewicz34, p. 225. A more accessible account can be found in Ajdukiewicz35.

the argument's index, and the whole 'fraction' would need to be equal to a proper part of the 'fraction'. Therefore the directives of Leśniewski's systems, by enforcing the concept of semantic categories, prevent certain contradictions which have been described as 'vicious circle paradoxes'[35].

Leśniewski says that although his concept has this effect of preserving the consistency of his systems, 'I would feel myself forced to accept it if I wanted to speak at all sensibly [*überhaupt mit Sinn*] even if there were no antinomies'[36]. In its formal consequences it resembles the 'theory of types', but the concept of semantic categories is more closely connected on its intuitive side with Aristotle's categories, with the parts of speech of traditional grammar, and with Husserl's 'categories of meaning' [*Bedeutungskategorien*][37]. Though the term 'semantic category' echoes or even translates Husserl's term, the concept appears closer to that of parts of speech[38].

1.4. Definitions in deductive systems

Many logicians unfamiliar with Leśniewski's work have particular difficulty in accepting the rôle of definitions in his systems. We must examine definitions briefly without attempting an extended defence of his views.

Non-primitive symbols often appear in symbolic expressions. Such symbols can be introduced informally or formally. Leśniewski uses non-primitive symbols of both kinds in his published works.

The particular quantifier '$\exists$' is an example of a symbol introduced informally into ontology and mereology. Statements in publications of Leśniewski and of his followers occasionally explain that in the 'official' systems of ontology and mereology there is no symbol corresponding to '$\exists$'. Informal expression of the type '$[\exists x].f(x)$' always correspond to expressions of the type '$\vdash\!\left(\llcorner x\lrcorner\ulcorner\vdash\!\left(f(x)\right)\urcorner\right)$' in the 'official' systems; the latter expressions contain only universal quantifiers and correspond to informal expressions of the type '$\sim[x].\sim f(x)$'[39]. We may say that the particular quantifier is informally defined by such statements. Neither the informal 'definition' nor the symbol it introduces actually belong to the system in question.

Some logicians would prefer to have all non-primitive symbols defined informally. Thus, for example, '$p\vee q$' may be defined to be an informal alternative for '$\sim p\supset q$'. But in Leśniewski's systems there are contexts in which '$\vee$' may appear, but not in an expression such as '$p\vee q$'. For example, we can define a functor 'Φ' which requires one argument in the semantic category of '$\supset$'. We can interpret '$\Phi<\supset>$', but how can we interpret '$\Phi<\vee>$'? One might argue that the latter expression could be interpreted by using the *definiens* of '$\vee$' in the *definiens* of 'Φ', but this does not work if we have a variable 'ϕ' in the same semantic category as the constant 'Φ': the expression '$\phi<\vee>$' is uninterpretable because there is no symbol or expression actually in the system which corresponds to '$\vee$'; there are

[35] WHITEHEADRUSSELL10, pp. 37–8 and 60–5.

[36] LEŚNIEWSKI29, p. 14.

[37] Ibid.

[38] LEJEWSKI65, p. 191.

[39] For example LEŚNIEWSKI30, p. 114.

only expressions which correspond to entire expressions of the type '$p \vee q$'. On the other hand, if we actually introduce the functor '$\vee$' into the system, it can legitimately appear in any appropriate context. Sometimes certain results can then be proved which are not provable unless '$\vee$' (or some other term) is formally introduced. For example, we might be able to prove that

$$[\exists f]\!:\![pq]\!:\!p.\supset.f(p,q)$$

Hence a system in which there is a formal definition of '$\vee$' can be stronger than a system 'in' which '$\vee$' is informally 'defined' but does not formally exist. Some definitions play an essential role in proving theses which do not contain the defined term and were meaningful before the definition was added to the system. These Łukasiewicz called *creative* definitions[40], though earlier writers used this phrase in a different sense. Creative definitions exist in standard systems of protothetic and ontology. There are no creative definitions in standard systems of mereology[41], and there are no creative definitions in computative protothetic.

There are two ways by which logicians have added a term formally to a deductive system as a new symbol: by adding new directives to the system, and by adding new theses to the system.

A new directive, such as a rule of replacement[42], can be added to a formalised system only if the directives permit it. That is, there must be a directive which allows the addition of new directives. This definition directive must allow us to determine unambiguously and in a finite number of steps whether it is legitimate to add any new (replacement) directive to the system. The added directive must allow us to determine unambiguously and in a finite number of steps whether it is legitimate to add any expression to the system as a new thesis. Moreover, the added directive must guarantee that the new term it introduces is completely defined, in the sense that every meaningful expression in the system which contains the term has a determinate meaning and does not violate the system's consistency. Finally, the definition directive must be complete, in the sense that it must enable us to add to the system replacement directives for all possible defined terms. At the present time we know of no way to formulate such a definition directive[43].

Alternatively, as in Leśniewski's standard systems, we can have a definition directive which permits us to add to the system new theses which serve as the definitions of additional terms. In the standard systems a definition, once it has been added, is treated just like any ordinary thesis, but in computative protothetic there are certain directives which refer to previous definitions in a special way.

Some logicians and mathematicians are horrified when they hear of such definitions. They have heard too often statements like this:

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[40] Łukasiewicz39; McCall67, p. 113; Łukasiewicz70, p. 275. This sense of 'creative' may be due to Leśniewski, who appears to use it in Łukasiewicz28, p. 178.

[41] This is not difficult to prove, but it does not seem to be widely known. Professor Lejewski has pointed out that when ontology is used as a basis for other theories, the axioms of these theories may allow us to prove theorems which do not contain any terms added to our vocabulary by the later theories, but which are not provable from the axiom system of ontology alone. In this sense one may say that, for example, the axiom system of mereology is 'creative' with respect to ontology.

[42] This terminology is used in Łukasiewicz29, p. 53.

[43] Leśniewski says this in Łukasiewicz28, p. 178.

> . . . a definition is, strictly speaking, no part of the subject in which it occurs. For a definition is concerned wholly with the symbols, not with what they symbolise. Moreover, it is not true or false, being the expression of a volition, not of a proposition. . . Theoretically, it is unnecessary ever to give a definition: we might always use the *definiens* instead, and thus wholly dispense with the *definiendum*. Thus although we employ definitions and do not define "definition", yet "definition" does not appear among our primitive ideas, because the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences[44].

When I am faced with a horrified mathematician, I am willing to grant for the sake of argument that the above quotation accurately describes 'definitions' in the system of Whitehead and Russell. We ignore that system and examine a quite different one, such as the standard system of protothetic or ontology, in which 'creative' definitions appear. Few mathematicians are so preoccupied with what they *want* to see that they cannot admit that the thesis in question is a definition, although it may be quite different from the 'definitions' in *Principia Mathematica* and other familiar works. After contemplating such a definition, most mathematicians are willing to allow me to state my position as follows:

There are some definitions which are, strictly speaking, part of the system in which they appear. They are not true or false or even meaningful relative to the portion of the system which precedes their introduction, but once they have been introduced they are meaningful and true[45]. It is often — perhaps even always — possible to use one definition rather than another, but there are circumstances in systems of this kind in which some definition is theoretically indispensable.

Łukasiewicz gives a simple example of such a system[46]. It is a subsystem of the 'theory of deduction' having propositional equivalence as its primitive term, substitution and detachment as its only directives, and free propositional variables. The axiom is a single thesis, $EEsEppEEsEppEEpqEErqEpr$ in Łukasiewicz's notation, or

$$s.{\equiv}.p{\equiv}p{:}{\equiv}{:}{:}s.{\equiv}.p{\equiv}p{:}{\equiv}{\therefore}p{\equiv}q.{\equiv}{:}r{\equiv}q.{\equiv}.p{\equiv}r$$

in the notation of Whitehead and Russell. Now the only theses we can prove in the system as described are substitutions of the axiom; we can prove that the axiom is 'undetachable'. But if we add to the system a definition of the traditional 'verum' functor, $EVpEpp$ in Łukasiewicz's notation, or $vr(p).{\equiv}.p{\equiv}p$ in Whitehead and Russell's, we can prove all classical equivalences in the resulting system.

1.5. Extensionality

We may say of two sentences that they are extensionally equivalent if they are both true or both false. We may say of two name-expressions that they are extensionally equivalent if any object named by either expression is named by the other; in the terminology of mediæval philosophers these names have the same extension. We may say of two expressions in some semantic category other than the categories of sentences or of names that they are extensionally equivalent if, whenever they are completed with extensionally equivalent arguments, they form extensionally equivalent functions.

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[44] WHITEHEADRUSSELL10, p. 11.

[45] Cf. Leśniewski's remarks reported in KOTARBIŃSKI24, p. 264.

[46] ŁUKASIEWICZ39; MCCALL67, pp. 113–5; ŁUKASIEWICZ70, pp. 275–7.

A deductive system can be described as extensional if, whenever two expressions are proven to be extensionally equivalent, then they can be proven to be mutually substitutable in all contexts. All of Leśniewski's deductive systems are extensional. In the standard systems of protothetic, ontology, and mereology there are directives which authorise us to add to the system a thesis guaranteeing substitutability for extensionally equivalent expressions belonging to any semantic category except that of sentences. The law of extensionality for sentences

$$[pq]\therefore p{\equiv}q.{\equiv}:[f]:f(p).{\equiv}.f(q)$$

can be proved without appealing to the extensionality directives[47]. In many of the non-standard systems of protothetic there is no directive for extensionality, but there are directives for verification which, together with the axioms and the other directives, ensure that theses can be proved which correspond to those which can be added to $\mathfrak{S}_5$ in accordance with the extensionality directive in that system.

We should note that the extensionality directives in Leśniewski's standard systems are closely related to the directives for definitions. Loosely speaking, we may say that if we can prove that two terms have 'the same' definition, then we can always substitute one term for the other in any context. In other words, the format of a definition gives a sufficient condition for the extensional equivalence of two terms.

Leśniewski originally added the extensionality directive to protothetic because it was deductively equivalent to the verification directive, but it could be formalised in a much simpler manner[48]. We know, however, that he was convinced that theses of extensionality were as true as any theses of classical logic[49]. It is very likely that the propositional extensionality directive was added to ontology at this time, and that the nominal extensionality directive was added to ontology soon after, when Leśniewski had reflected on it and decided that it was valid[50].

Those logicians who object to extensional logic rarely discuss the extensionality of names; they seem to be most concerned with the so-called 'intensional' functors, such as 'knows that' or 'believes that', functors whose arguments they analyse as sentences. Leśniewski was able to analyse sentences involving such functors in an extensional manner. In the sentence '*A* believes "p"' the expression '"p"' is a name for sentences, a name in which there is no variable 'p', despite the fact that is seems to appear there. The quotes around 'p' are not a smuggled-in intensional functor; rather the entire expression '"p"' is an informal symbol for some name whose formal definition may not be quite the same from one context to another.

We know of no 'intensional' functor which cannot be analysed as an extensional functor in such a fashion[51]. Professor Lejewski has recently demonstrated that, in questions involving extensionality and 'singular' names, the 'intensional' analysis of 'believes' and related words is inconsistent with certain logical principles which hitherto have not yet been

47 Leśniewski29, p. 30.

48 Leśniewski29, pp. 41–4.

49 Leśniewski29, pp. 30, 42–3.

50 This addition is mentioned in Sobociński34, p. 160.

51 Cf. Tarski56, p. 8.

called into question[52]. Moreover the extensional analyses seem in many ways intuitively more satisfactory than the 'intensional' ones. Those of us who follow Leśniewski therefore feel that extensionality is a valuable feature of his system, since it encourages us to avoid incorrect semantic analysis of apparently 'intensional' terms. Furthermore we may expect that few logicians will agree on the correctness of any given deductive system which appears to contain 'intensional' functors, since an inadequate semantic analysis will betray itself by making readers uneasy about some of the system's theses.

There are, of course, many logicians and philosophers who have claimed that certain theses of classical two-valued extensional logic make them uneasy. Leśniewski too was suspicious of classical logic for several years, but he decided in the end that he had been misled by the sloppily written commentaries in which many logicians surround their systems, as if they wish to discourage readers from understanding what they have written[53].

1.6. Computative protothetic

Between 1924 and 1934 Leśniewski constructed several systems of protothetic which do not contain substitution, detachment, or extensionality among their directives. He describes these systems briefly, characterising their style of inference as 'automatic verification'[54], a style suggested to him by the article Łukasiewicz20. Leśniewski and his students referred to these systems as '*prototetyka obliczeniowa*'[55], which Sobociński later translated as 'calculation system[s] of protothetic'[56], and as 'systems of computable protothetic'[57]. Leśniewski used them to prove that the 'standard' system of protothetic $\mathfrak{S}_5$ is consistent and complete[58]. The term 'computative protothetic' has been used by later writers[59].

Loosely speaking, systems of computative protothetic are based on a verification directive similar to Peirce's 0–1 method but extended to every semantic category which can be introduced into the system. Since there is no limit to the number of matrices potentially required to verify theses, the 'matrices' are not given in the usual tabular form, nor do they exist, as it were, 'outside' of the system. Instead the information conveyed by matrices in other systems is expressed by various theses which actually belong to this system. Verification and rejection can take place in a new semantic category as soon as certain necessary theses have been proved.

The systems of computative protothetic appear to have been designed in such a manner that any meaningful expression can be proved or disproved in only one way. This means that their consistency and completeness are relatively simple to prove. Among the disadvantages of this approach are its inflexibility and its inability to be extended.

The 'standard' systems of protothetic, ontology, and mereology all involve us in very lengthy deductions if we prove theses step by step, but most complex procedures will actually

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52 Lejewski81, pp. 218–21.
53 Cf. Leśniewski27, p. 170.
54 Leśniewski38, p. 35.
55 Sobociński54, p. 18.
56 Sobociński49, p. 14.
57 Sobociński60, p. 54.
58 Sobociński54, p. 18, and Sobociński60, p. 56.
59 E.g., Luschei62, pp. 39 and 153.

require fewer and fewer steps as the system develops. For example, in a system of protothetic with the directives of $\mathfrak{S}_5$ and based on Sobociński's axiom A_p, within a few steps of the beginning of the system we are able to reverse any equivalence, and this will take up to 12 applications of the directives in the worst possible cases. After many deductions we can prove a thesis which allows us to reverse any equivalence in at most three applications of the directives. In computative protothetic there are no such short cuts.

A 'standard' system of protothetic can be extended to form the basis for a system of ontology simply by allowing its directives to regard the axiom (or axioms) of ontology as a thesis. In standard protothetic a sentence consisting of a 'verb' with two 'nouns' cannot be substituted for a propositional variable because such an expression has no meaning in the system. When the axiom of ontology is added, the sentence becomes meaningful and the substitution is legitimate. But computative protothetic has no substitution directive; it requires all complex expressions to be built up piece by piece using the definitions of all constants contained in the expression or belonging to the semantic category of any variable appearing in the expression. In effect this means that we could not use the ordinary axioms of ontology. We would need to replace them with a very large number of simple axioms containing no variables, and if we wished the theory to apply to an infinite number of objects, as standard ontology may, we should need an infinite number of axioms. Computative protothetic is equivalent to $\mathfrak{S}_5$, but it cannot be used as a basis for general ontology or for any but the very simplest of theories.

Leśniewski warns us that, although he formalised computative protothetic completely, his published description of one system's directives is a 'brief, sketchy, inexact' outline, written 'without observing the necessary precautions'[60]. In particular he points out that he has not 'effectively' formulated the 'schema' for defining the 'basic constants' which are required before applying the verification directive[61]. It is most unusual for Leśniewski to present a system in this informal and incomplete fashion. It is likely that he did so mainly because of the importance of this alternative approach, that is, because he felt that the computative systems provide considerable insight into the nature of protothetic.

Computative protothetic is of some interest in its own right: its axiom systems are very simple, its directives are unusual, and its deductive structure is quite easy to grasp. The study of such systems can provide an introduction to Leśniewski's theories and to his metalogical methodology, and it can lead to a deeper insight into protothetic in general[62].

[60] Leśniewski38, p. 36.
[61] Leśniewski38, p. 38.
[62] Cf. Rickey77, pp. 413–4.

2. The Authentic Symbolism

The 'official' systems of protothetic, ontology, and mereology are written using a special symbolism devised by Leśniewski, which he calls the 'authentic' symbolism of his systems[1]. In the present work we must use this symbolism rather than any of the more familiar notations because it makes the task of stating directives very much easier. (In fact it is difficult to see how the directives could be formulated at all if we used, for example, Whitehead and Russell's notation.) Moreover, we shall want to compare our directives with those of $\mathfrak{S}_5$, which are formulated with the authentic symbolism in mind, and the task of comparing the two systems in detail becomes very burdensome if the directive formulations lack a common basis.

Clearly the primary goal in the design of the authentic symbolism was simplifying the directives of the standard systems of protothetic and ontology[2]. Leśniewski felt that unnecessarily complex directives would form a significant barrier cutting his system off from future readers[3]. But although he designed his notation to make the directives simpler and clearer, it also proved to be able to express the theses of the system very clearly. A number of conventions help readers to spot at a glance the groupings of parentheses and of the indicators of quantifier scopes. The result is not as compact (or as easy to print) as the bracket-free notation devised in 1924 by Łukasiewicz[4], but it has the advantage of not requiring the reader to distinguish semantic categories from each other by the alphabet used for their variables. Leśniewski claimed that it was the clearest symbolism he knew, as opposed to the bracket-free notation, which he said was the simplest but not the clearest [*durchsichtigste*] notation he knew[5].

The notations of Leśniewski and Łukasiewicz resemble each other in that all functors come before their arguments; they are both what is often called 'prefix' notations. It was apparently Leon Chwistek who suggested this[6], presumably in 1920, when he convinced Leśniewski to start using 'logical symbols' instead of ordinary words in theses of his systems[7].

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[1] E. g., Leśniewski29, p. 44, and Leśniewski38, p. 5. Leśniewski never actually published in any of his works a thesis of mereology expressed in the authentic symbolism, but his description of the terminological explanations associated with mereology in Leśniewski29, pp. 68–9, makes it clear to anyone familiar with this section of his work that the language to which they apply is expressed in the authentic symbolism.

[2] Professor Lejewski reports that Leśniewski compared the Peano-Whitehead-Russell symbolism of his lectures and of many of his publications to casual dress, and compared his authentic symbolism to formal clothing, which, he said, it was appropriate to wear on special occasions.

[3] Leśniewski29, p. 37.

[4] Łukasiewicz29, pp. 26–31 and 38–42. Cf. also Łukasiewicz25 and Leśniewski's remarks reported there.

[5] Cf. Leśniewski31, p. 291, where the 'clearest' symbolism is obviously Leśniewski's own.

[6] Luschei62, p. 107.

[7] Leśniewski31a, p. 154.

2.1. Terminology

The sentences which constitute Leśniewski's deductive systems are themselves made up of basic elements called *words*. A word is an expression no part of which is an expression. The following expressions are examples of words: ‘man’, ‘word’, ‘p’, ‘ϕ’, ‘$\llcorner$’, ‘$\urcorner$’, ‘(’, ‘}’. The following expressions are collections of words, but they are not words: ‘the man’, ‘(p)’, ‘$f\llcorner)$word’. The three expressions just cited consist of two, three, and four words respectively. The following thesis of protothetic consists of fifty-four words:

$$\llcorner pq \lrcorner \ulcorner \phi\Bigg(\phi(pq)\llcorner f \lrcorner \ulcorner \phi\Bigg(f\Big(pf(p\llcorner u \lrcorner \ulcorner u \urcorner)\Big)\llcorner r \lrcorner \ulcorner \phi\Big(f(qr)\phi(qp)\Big)\urcorner\Bigg)\urcorner\Bigg)\urcorner$$

A letter or index which is merely part of a word is not a word. An expression containing two or more words is not a word.

An *expression* is a collection of successive words. Every word is an expression. The collection of any number of consecutive words of an expression is an expression. The collection consisting of the first, third, and fourth words of some expression is not an expression because it has a ‘hole’ in it. Every expression consists of a finite number of words. If there were an object consisting of an infinite number of words, it would not be an expression.

When two words or expressions have the same shape, they are said to be *equiform.* The fourth word of the thesis A_p cited above is equiform with the fifteenth word of the same thesis. The expression consisting of the second and third words of A_p is equiform with the expression consisting of the tenth and eleventh words of the same thesis. The word ‘$\Bigg($’ is equiform with the word ‘(’: Leśniewski allows equiform parentheses to vary in size, so as to improve the reader's ability to spot the structure of an expression without having to count the parentheses. The word ‘(’ is not equiform with either of the words ‘{’ or ‘[’: parentheses have different shapes for certain special purposes, so it is not possible to use them as typographical variants. The word ‘$^{\ulcorner}$’ is equiform with the word ‘$\ulcorner$’: Leśniewski allows these words, which indicate the scope of quantifiers, to vary in height in order that expressions may be more perspicuous. Note that none of the words ‘$\llcorner$’, ‘$\lrcorner$’, ‘$\ulcorner$’, and ‘$\urcorner$’, is a parenthesis[8].

The word ‘term’ is defined in the terminological explanations[9], but it is also useful in the present, informal context. A *term* is any word which is neither a parenthesis nor equiform with one of the four quantifier indicators. In Leśniewski's standard systems any term may be used as a constant or as a variable anywhere in the system, even in two parts of the same expression[10]. Two constants in different semantic categories may have the same shape, as there is no possibility of confusing them[11]. A variable is simply a word bound by a quantifier. Moreover, there is no particular shape officially associated with variables of one or another semantic category. There are conventions that sentence variables are taken from the series ‘p’, ‘q’, ‘r’, ..., that name variables are taken from the series ‘A’, ‘B’, ‘C’, ..., if they must be ‘singular’ to make some part of the containing sentence true, and

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[8] See Leśniewski29, pp. 61–2, for Leśniewski's comments on these terms.

[9] Leśniewski29, p. 63.

[10] Leśniewski29, p. 76.

[11] E. g., Sobociński34, pp. 152 and 159.

from the series ‘a’, ‘b’, ‘c’, …, if they may be ‘singular’ or ‘plural’ or ‘empty’, that variable functors are taken from the series ‘f’, ‘g’, ‘h’, …, if they have arguments in the groups just mentioned, and from the series ‘ϕ’, ‘χ’, ‘ψ’, …, when they have arguments ‘f’, ‘g’, ‘h’, …. These conventions have no ‘official’ character; they exist only to hint to someone who reads the thesis what it is intended to mean.

The word ‘function’ is used in a sense slightly different from that in which Frege used it: a function consists of a term and one or more pairs of parentheses enclosing arguments. That is, the parentheses and all of the arguments are part of the function. That part of a function which precedes its final group of arguments and their enclosing parentheses is called the ‘function sign’ or ‘functor’[12]. The ability of a function to have more than one bracketed expression completing it is characteristic of Leśniewski’s systems. Functions of this kind are sometimes called ‘many-link’ functions. The term which is their main functor belongs to a semantic category whose ‘index’ is a ‘fraction’ which has another, smaller ‘fraction’ as its numerator. Such a functor, when it is followed by appropriate arguments enclosed in parentheses, becomes a function which is itself the functor of a larger function. Leśniewski describes many-link functions as the result of generalising from certain functor-forming functions in *Principia Mathematica*[13].

The directives of the systems do not allow us to prove any thesis in which the expression which is under the scope of a universal quantifier is itself a generalisation[14]. That is, where one might in the systems of some other logicians have expressions such as the following

$$\llcorner ab\ldots\lrcorner \ulcorner \llcorner kl\ldots\lrcorner \ulcorner f(ab\ldots kl\ldots)\urcorner\urcorner$$

which in more traditional symbolism would appear as

$$[ab\ldots]\!:\![kl\ldots].f(ab\ldots kl\ldots)$$

in Leśniewski’s systems we are allowed to have only the corresponding expressions

$$\llcorner ab\ldots kl\ldots\lrcorner \ulcorner f(ab\ldots kl\ldots)\urcorner$$

2.2. The syntax of Leśniewski’s systems

It is difficult to discriminate in the metatheory of Leśniewski’s systems between what some contemporary writers would call *syntax* and *semantics*. The difficulty arises at least in part because aspects of both sides of this distinction are found throughout the terminological explanations. Because those explanations appear rather formidable, many readers simply ignore the ‘official’ descriptions of the language of Leśniewski’s systems. I therefore feel it may be useful to give a simple, conventional description of what others might call the ‘syntax’ of the authentic symbolism; perhaps the simplicity of this explanation may encourage some timid souls to wade through the more accurate account. This description has no official character and does not resemble anything written by Leśniewski himself.

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12 The term ‘functor’ was invented by Tadeusz Kotarbiński; cf. Tarski56, p. 161.

13 Leśniewski29, p. 66, refers to $x\{\mathrm{Cnv}‘(\mathrm{P}\dot{\cap}\mathrm{Q})\}y$ from WhiteheadRussell10, p. 239.

14 Leśniewski29, p. 77.

The first part of the following description uses a variety of the Backus-Naur Form (BNF) invented in 1963 for describing 'context-free' grammars, and now widely used in computing circles. The expression to the left of the '=' is described by the expression to the right. Symbols in curly braces '{' and '}' may be omitted, or they may be repeated any number of times. Symbols separated by '$\vee$' are alternatives any one of which may be chosen. Words in quotation marks are equiform with words in the expression being described. The terminology is approximately that of the terminological explanations, which will be defined precisely in a later chapter.

> genl = '$\llcorner$' trm {trm} '$\lrcorner$' '$\ulcorner$' essnt '$\urcorner$'.
> essnt = trm $\vee$ fnct.
> fnct = trm prntm {prntm}.
> prntm = left-parenthesis arg {arg} right-parenthesis.
> arg = trm $\vee$ fnct $\vee$ genl.

The above 'syntactic' description of the authentic symbolism omits a very large number of restrictions imposed by the terminological explanations. I shall now summarise a few of these restrictions.

In very loose terms, a meaningful expression is a term, function, or generalisation which belongs to the semantic category of sentences. An expression can only be meaningful relative to a particular stage of development of a deductive system, within which all of its constants are primitive terms or have been defined[15].

Every term in the quantifier of a generalisation must bind at least one variable in the quantified part (essnt) of the generalisation. That is, there are no 'vacuous' quantifiers[16].

There does not exist in the authentic symbolism a single shape of parenthesis. Instead there are an unlimited number of possible shapes of parenthesis. These are paired into 'left' and 'right' forms which are described as symmetrical (prntsym). A pair of symmetrical parentheses are never equiform with each other, but right parentheses are equiform with each other if they are symmetrical to equiform left parentheses.

Like the constants in different semantic categories, equiform parentheses may have an unlimited number of 'semantic' functions. Two equiform parentheses will have the same 'semantic' function if, and only if, they begin bracketed expressions which contain the same number of arguments. In that case, the functions which the bracketed expressions terminate belong to the same semantic category, and the corresponding arguments must also belong to the same semantic categories. The directives are formulated in such a way that it is forbidden for there to be more than one way of representing such a function; that is, if two functions belong to the same semantic category, and if their final bracketed expressions have the same number of arguments belonging respectively to the same semantic categories, then the left parentheses beginning the final bracketed expressions must be equiform.

Only terms, functions, and generalisations in the authentic symbolism are defined as belonging to a semantic category. Loosely speaking, the semantic category of an expression is determined in two ways: it is determined from 'outside' by being a thesis, the

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[15] A terminological explanation giving the precise definition of meaningful expressions in standard protothetic appears in Leśniewski31, pp. 301–2.

[16] Leśniewski31, p. 301.

nucleus (essnt) of a generalisation, such-and-such an argument of a bracketed expression, or a functor followed by a particular kind of bracketed expression; it is determined from 'inside' by being a generalisation, or a variable bound to a related variable in an appropriate semantic category, or an unbound term equiform with a defined constant in an appropriate semantic category, or a function whose final bracketed expression determines by its number of arguments and by the shape of its parentheses the category of the function. The directives ensure that the 'inside' and 'outside' determinations of the semantic category of an expression in any thesis always agree with each other[17].

2.3. The basic outlines for constants

The forms used for constants in the authentic symbolism of protothetic are purely conventional and have no 'official' character. Nevertheless it is useful to know the system, and so to be able to recognise new constants when they appear, and to see what their definer intends them to mean.

The conventions specify 'basic outlines' for constants in three semantic categories[18]: those with the indices 's', '$\frac{\text{s}}{\text{s}}$', and '$\frac{\text{s}}{\text{s s}}$'. The two outlines in the sentence category are 'Λ' and 'V' for 'false' and 'true' respectively. The four outlines for '$\frac{\text{s}}{\text{s}}$' functors are '—', '⊣', '⊢', and '⊢⊣'; in these the vertical bar on the left is present if, and only if, the function is true when its argument is false; the vertical bar on the right is present if, and only if, the function is true when its argument is true. The '$\frac{\text{s}}{\text{s s}}$' functors have sixteen basic outlines: '○', '⫯', '⫰', '○-', '-○', 'ϕ', '⫯-', '-⫯', '⫰-', '-⫰', '-○-', 'ϕ-', '-ϕ', '-⫯-', '-⫰-', and '-ϕ-'. In these the bottom arm occurs if, and only if, the function is true when both arguments are true; the top arm occurs if, and only if, the function is true when both arguments are false; the left arm occurs if, and only if, the function is true when its first argument is true and the second is false; the right arm occurs if, and only if, the function is true when its first argument is false and its second is true[19].

Thus the following table lists some of the more common correspondences between expressions that might be found in *Principia Mathematica* or in works which more or less follow the same conventions, and those that might appear in Leśniewski's authentic symbolism:

Principia	Leśniewski	*Principia*	Leśniewski
$\sim p$	⊢(p)	$p.\vert.q$	-⫰-(pq)
$p.\supset.q$	ϕ-(pq)	$p.\downarrow.q$	⫰(pq)
$p.\vee.q$	-⫯-(pq)	$(p).f(p)$	$\llcorner p \lrcorner \ulcorner f(p) \urcorner$
$p.q$	⫯(pq)	$(p):(q).f(p,q)$	$\llcorner pq \lrcorner \ulcorner f(pq) \urcorner$
$p.\equiv.q$	ϕ(pq)	$(\exists p).f(p)$	⊢$\Big(\llcorner p \lrcorner \ulcorner$⊢$\big(f(p)\big)\urcorner\Big)$

In addition to the basic outlines, constants may have an index. For example, the three constants '⊢', '$\underset{1}{\vdash}$', and '$\underset{2}{\vdash}$' are not equiform, but have the same 'truth conditions'.

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[17] Leśniewski31, pp. 301–2.

[18] See Leśniewski38. pp. 21–3.

[19] Note that the account of these symbols in Quine40 is incorrect.

In standard systems of protothetic such synonymous constants are introduced by different definitions and usually have only temporary interest.

The authentic symbolism is described accurately and officially in the terminological explanations of protothetic and of ontology. Sensitive use of this symbolism requires us to conform to a large number of conventions and redundant features, but most of these contribute significantly to the perspicuity of the expressions constructed in the symbolism.

3. The History of Protothetic

An outline of the history of protothetic provides one perspective on the theory. This helps to explain how it extends the traditional 'theory of deduction' and why these extensions were added.

3.1. The foundations of mathematics

In 1911 Leśniewski learned of the existence of symbolic logic and of Russell's antinomy concerning the 'class of the classes which are not elements of themselves'[1]. He was distressed by this antinomy, and he believed that all attempts of mathematicians to solve it had strayed rather far from the intuitive basis of the problem:

> The only method of effectively 'solving' the 'antinomies' is the method of an intuitive undermining of the inferences or presuppositions which together contribute to the contradiction. A mathematics separated from intuition contains no effective medicines for the infirmities of intuition[2].

Leśniewski's first step was to become familiar with symbolic logic. He says that he spent four years[3] gradually overcoming his initial aversion to this discipline, which he attributed to the 'hazy, ambiguous commentaries which workers in this field have provided for it'[4]. After studying the systems of others, he began to produce his own deductive theories in the reverse order of their logical dependence, publishing in 1916 his first work on mereology[5], constructing in 1920 the first axiom for ontology[6], and the first system of protothetic two years later[7].

In the period between 1916 and 1922 the style of Leśniewski's work changed markedly. In 1916 he wrote proofs in ordinary language, supplemented by variables and a few technical terms; his style was formal but not formalised, and his 'natural deduction' had rather a Euclidean flavour. By 1922 he was writing proofs entirely in logical symbols, and though he used natural deduction for some of them, others were constructed using substitution and detachment. These systems were highly formalised, with dozens of technical terms defined by terminological explanations written in ordinary language with the help of variables and technical terms. Leśniewski did not believe that his formalism made his systems more remote from his 'logical intuitions'. He saw 'no contradiction in wishing to maintain that I practise an apparently radical formalism' despite being 'an obdurate intuitionist'[8]. In his deductive system he 'entertained a series of thoughts expressed in a series of sentences', deriving one from another by inferences which he considered 'binding'; he knew no method better than formalising them for acquainting a reader with his 'logical intuitions'[9].

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[1] Leśniewski27, p. 169.
[2] Leśniewski27, p. 167.
[3] Leśniewski27, p. 181.
[4] Leśniewski27, p. 170.
[5] Leśniewski16.
[6] Leśniewski30, p. 114.
[7] Leśniewski29, pp. 36–7.
[8] Leśniewski29, p. 78.
[9] Ibid.

After he had created protothetic in 1922, Leśniewski's system of foundations for mathematics was essentially complete. He insisted that it was only one of the many possible foundations of mathematics, and he cautiously admitted that he was satisfied with it 'for the time being' [*narazie*][10]. He spent the remaining seventeen years of his life studying and attempting to improve his three deductive theories, concentrating for much of that time on simplifying the axiom systems and the directives.

3.2. General characteristics

Leśniewski's critical study of earlier deductive systems led him to a number of conclusions which shaped protothetic significantly.

It was not clear to him what signs of assertion (and signs of rejection) mean, and whether or not they are actually part of the theses of deductive systems[11], so that before 1918 he decided simply to ignore them[12]. Consequently protothetic and Leśniewski's other theories have no such symbols.

By 1920 he concluded that there was no need for 'real' variables[13]; hence in protothetic all variables must be bound explicitly by universal quantifiers. Leśniewski did not introduce particular quantifiers into his 'official' systems. He reasoned that there are not just two sorts of quantifier, particular and universal, but an unlimited number; for example, there are the quantifiers 'for at least two', 'for at most five', and 'for between three and six'[14]. He could see no way to introduce all possible quantifiers into his systems with appropriate formalisation, and decided in the end that it was inappropriate to introduce more than one sort of quantifier without introducing all of them[15].

Sheffer in 1912 showed that the 'theory of deduction' could be based on a single primitive term instead of two, such as Frege (implication and negation) and Whitehead and Russell (alternation and negation) had used[16]. In 1916 Nicod constructed an axiom system for one of Sheffer's terms[17]. Both Sheffer and Nicod make use of a special symbol for definitional equivalence and special rules of replacement for defined terms. Leśniewski believed that definitions are in fact part of deductive systems, and must be expressed using the primitive term or terms of the system. Therefore in 1921 he remarked that it was difficult to accept that the systems of Sheffer and of Nicod are based on a single primitive term. This could be remedied if the equivalences were expressed using the primitive term

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[10] Leśniewski27, p. 168.

[11] Leśniewski27, pp. 170–5.

[12] Leśniewski27, p. 181.

[13] Łukasiewicz20, p. 189; Leśniewski29, p. 31.

[14] Leśniewski never discussed such quantifiers in print, but he apparently believed that they resemble the numbers of ordinary language more closely than do the numerical functors which can be defined in ontology. Professor Lejewski learned of this from Bolesław Sobociński, Leśniewski's student and closest collaborator.

[15] Lejewski56, p. 191.

[16] Cf. Sheffer13. In 1880 C. S. Peirce had actually discovered that one of Sheffer's functors had this property, but this was not known until recently.

[17] Nicod20.

of the appropriate system[18]. For example, as he suggested in 1933, since it is easy to prove that

$$[pq]{::}p{\equiv}q.{\equiv}{.\!\cdot\!.}p|q.|{:}p|p.|.q|q$$

expressions of the type ‘$p|q.|{:}p|p.|.q|q$’ should be used for expressing definitions in the system of Nicod[19].

Definitions expressed in such a fashion are obviously not as simple or as intuitive as definitions expressed as equivalences, so that a system based on equivalence as a primitive term would be more attractive than a system based on the ‘stroke’ functor[20]. Such a system would not be very satisfactory, however, if it was not strong enough to allow all possible functors to be defined. In 1922 Alfred Tarski, who was then completing his doctorate under Leśniewski’s supervision, discovered that conjunction could be defined in terms of equivalence[21]. At the time Tarski found two possible definitions:

$$[pq]{::}p.q.{\equiv}{.\!\cdot\!.}[f]{.\!\cdot\!.}p.{\equiv}{:}[r].p{\equiv}f(r).{\equiv}.[r].q{\equiv}f(r)$$
$$[pq]{.\!\cdot\!.}p.q.{\equiv}{:}[f]{:}p.{\equiv}.f(p){\equiv}f(q)$$

According to Tarski the first definition is true in all systems, while the second is true in those systems in which the law of extensionality for expressions in the semantic category of sentences can be proved:

$$[fpq]{:}p{\equiv}q.f(p).{\supset}.f(q)$$

Since, for example, the thesis $[p]{.\!\cdot\!.}{\sim}p.{\equiv}{:}p.{\equiv}.[u].u$ can easily serve as the definition of negation, a system based on equivalence as its only primitive term can be functionally complete.

Leśniewski decided that deductive systems which are based on a single primitive term are superior to systems based on more than one primitive term; such systems are not logically better but they are æsthetically more satisfying[22]. The standard systems of protothetic, ontology, and mereology are each based on a single primitive term. Systems of computative protothetic, for reasons which will be explained later, must be based on at least two primitive terms.

In 1922, the year in which Tarski discovered how to define conjunction in terms of equivalence, Leśniewski outlined his notion of semantic categories[23]. With this it became possible to construct an elegant replacement for the ‘theory of deduction’ which would satisfy Leśniewski in all respects.

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[18] Leśniewski29, pp. 9–11.

[19] Leśniewski38, pp. 16–17.

[20] Leśniewski later observed that there is another primitive term which is nearly as elegant as ‘ϕ’, namely ‘-o-’, the functor of inequivalence. I learned of this unpublished remark from Professor Lejewski.

[21] Leśniewski29, pp. 11–13; Tarski56, pp. 2, 7–8.

[22] Sobociński56, p. 55.

[23] Leśniewski29, p. 14.

3.3. The theory of pure equivalence

In 1922 Leśniewski began to create protothetic by constructing a deductive system in which he could prove any of the 'pure equivalences' which can be proved in the ordinary 'theory of deduction'. This first subsystem of protothetic has directives only for substitution and detachment and is based on the following two axioms[24]:

A1	$p{\equiv}r.{\equiv}.q{\equiv}p{:}{\equiv}.r{\equiv}q$	$(EEEprEqpErq)$
A2	$p.{\equiv}.q{\equiv}r{:}{\equiv}{:}p{\equiv}q.{\equiv}.r$	$(EEpEqrEEpqr)$

Leśniewski referred to this subsystem of protothetic as system $\mathfrak{S}$[25]. The thesis *A2* is the law of associativity for equivalence, which had been proved before 1922 by Łukasiewicz[26].

In 1929 Leśniewski published a proof of the completeness of system $\mathfrak{S}$[27]. In that proof he establishes the consistency of $\mathfrak{S}$ relative to the consistency of the classical 'theory of deduction', but he also establishes that theses can be added to $\mathfrak{S}$ if, and only if, the number of equiform variables of each shape is *even*[28]. Łukasiewicz later observed that this gives us a simple structural proof of the consistency of $\mathfrak{S}$[29].

The later development of system $\mathfrak{S}$ is of some interest in its own right and has influenced the later development of protothetic. Between 1925 and 1930 Mordchaj Wajsberg discovered several axiom systems for $\mathfrak{S}$, including the first two single axioms[30]:

W1a	$p.{\equiv}.q{\equiv}r{:}{\equiv}{:}p{\equiv}q.{\equiv}.r$	$(EEpEqrEEpqr)$
W1b	$p{\equiv}q.{\equiv}.q{\equiv}p$	$(EEpqEqp)$
W2a	$p.{\equiv}.q{\equiv}r{:}{\equiv}{:}r.{\equiv}.q{\equiv}p$	$(EEpEqrErEqp)$
W2b	$p{\equiv}p.{\equiv}.p{:}{\equiv}p$	$(EEEpppp)$
W3	$p{\equiv}q.{\equiv}.r{:}{\equiv}.s{\therefore}{\equiv}{\therefore}s.{\equiv}{:}p.{\equiv}.q{\equiv}r$	$(EEEEpqrsEsEpEqr)$
W4	$p.{\equiv}.q{\equiv}r{:}{\equiv}{:}r{\equiv}s.{\equiv}.s{\therefore}{\equiv}.p{\equiv}q$	$(EEEpEqrEErssEpq)$

In 1926, searching for an axiom which resembled the law of extensionality for sentences, but which would be adequate for system $\mathfrak{S}$ as well, Wajsberg discovered the following axiom; it is not a thesis of system $\mathfrak{S}$, but given certain additional directives, all theses of system $\mathfrak{S}$ can be proved from it[31]:

W5 $\quad p{\equiv}q.{\equiv}{\therefore}[g]{\therefore}g(r{\equiv}s.{\equiv}.t,q){:}{\equiv}{:}g(s{\equiv}t.{\equiv}.r,p)$

$(EEpq\Pi\delta E\delta EErstq\delta EEstrp)$

Once the first single axiom for $\mathfrak{S}$ had been discovered, several researchers began to search for others. Between 1927 and 1932 six more single axioms were discovered, all the same length as *W3* and *W4*. Of these *W7* was discovered by Jerachmiel Bryman, *W6* and *W8* by Łukasiewicz, and *W9*, *W10*, and *W11* by Bolesław Sobociński[32]:

[24] Leśniewski29, pp. 15–16.

[25] Leśniewski29, pp. 15–16.

[26] Leśniewski29, p. 16; Tarski56, p. 4.

[27] Leśniewski29, pp. 16–30.

[28] Leśniewski29, pp. 26 and 29.

[29] Łukasiewicz39; McCall67, pp. 107–8; Łukasiewicz70, pp. 269–70.

[30] Wajsberg37; McCall67, pp. 314–6.

[31] Leśniewski38, p. 29.

[32] Sobociński32, pp. 186–7.

W6	$s.{\equiv}{:}p.{\equiv}.q{\equiv}r{\therefore}{\equiv}{:}p{\equiv}q.{\equiv}.r{\equiv}s$	($EEsEpEqrEEpqErs$)
W7	$p.{\equiv}.q{\equiv}r{:}{\equiv}{\therefore}q.{\equiv}.s{\equiv}r{:}{\equiv}.s{\equiv}p$	($EEpEqrEEqEsrEsp$)
W8	$p.{\equiv}.q{\equiv}r{:}{\equiv}{\therefore}q.{\equiv}.r{\equiv}s{:}{\equiv}.s{\equiv}p$	($EEpEqrEEqErsEsp$)
W9	$p.{\equiv}.q{\equiv}r{:}{\equiv}{\therefore}p.{\equiv}.r{\equiv}s{:}{\equiv}.s{\equiv}q$	($EEpEqrEEpErsEsq$)
W10	$p.{\equiv}.q{\equiv}r{:}{\equiv}{\therefore}p.{\equiv}.s{\equiv}r{:}{\equiv}.s{\equiv}q$	($EEpEqrEEpEsrEsq$)
W11	$p.{\equiv}.q{\equiv}r{:}{\equiv}{\therefore}p.{\equiv}.r{\equiv}s{:}{\equiv}.q{\equiv}s$	($EEpEqrEEpErsEqs$)

In 1933 Łukasiewicz discovered three single axioms shorter than any others known at that time[33]:

W12	$p{\equiv}q.{\equiv}{:}r{\equiv}q.{\equiv}.p{\equiv}r$	($EEpqEErqEpr$)
W13	$p{\equiv}q.{\equiv}{:}p{\equiv}r.{\equiv}.r{\equiv}q$	($EEpqEEprErq$)
W14	$p{\equiv}q.{\equiv}{:}r{\equiv}p.{\equiv}.q{\equiv}r$	($EEpqEErpEqr$)

At the same time he discovered that no single axiom for $\mathfrak{S}$ is shorter than any of these[34]. Łukasiewicz was mistakenly believed to have shown that there were no other axioms of this length for $\mathfrak{S}$[35], but C. A. Meredith found two others in 1951:

W15	$p{\equiv}q.{\equiv}.r{:}{\equiv}{:}q.{\equiv}.r{\equiv}p$	($EEEpqrEqErp$)
W16	$p.{\equiv}{\therefore}q.{\equiv}.p{\equiv}r{:}{\equiv}.r{\equiv}q$	($EpEEqEprErq$)

Meredith later discovered six further axioms of the same length[36]:

W17	$p.{\equiv}{\therefore}q.{\equiv}.r{\equiv}p{:}{\equiv}.r{\equiv}q$	($EpEEqErpErq$)
W18	$p.{\equiv}.q{\equiv}r{:}{\equiv}{:}r.{\equiv}.p{\equiv}q$	($EEpEqrErEpq$)
W19	$p{\equiv}q.{\equiv}{\therefore}r.{\equiv}{:}q{\equiv}r.{\equiv}.p$	($EEpqErEEqrp$)
W20	$p{\equiv}q.{\equiv}{\therefore}r.{\equiv}{:}r{\equiv}q.{\equiv}.p$	($EEpqErEErqp$)
W21	$p.{\equiv}.q{\equiv}r{:}{\equiv}.r{\therefore}{\equiv}.q{\equiv}p$	($EEEpEqrrEqp$)
W22	$p.{\equiv}.q{\equiv}r{:}{\equiv}.q{\therefore}{\equiv}.r{\equiv}p$	($EEEpEqrqErp$)

At least three completeness proofs have been published for system $\mathfrak{S}$: by Leśniewski[37], Mihailescu[38], and Łukasiewicz[39].

[33] Łukasiewicz39; McCall67, pp. 93 and 96–9; Łukasiewicz70, 255 and 258–61.
[34] Łukasiewicz39; McCall67, pp. 108–12; Łukasiewicz70, pp. 270–5.
[35] Sobociński49, p. 10.
[36] Meredith63, p. 185; see also Peterson76 and Kalman78.
[37] Leśniewski29, pp. 16–30.
[38] Mihailescu37.
[39] Łukasiewicz39; McCall67, pp. 99–104; Łukasiewicz70, pp. 261–6.

3.4. Equivalences plus bivalence

Leśniewski next considered what axioms and directives he needed to add to system $\mathfrak{S}$ to obtain the complete 'propositional calculus', including all terms that are ordinarily defined in it, together with the law of extensionality for sentences[40]:

$$[fpq] \therefore p \equiv q . \supset : f(p) . \equiv . f(q)$$

He promised to discuss this thesis at length in a later instalment of '*Grundzüge*', paying particular attention to the doubts which others might have about it, but he never kept this promise. Instead we have only the bare statement that

> I intended to construct a system in which, among others, just such a thesis would be provable, because from 1922 to the present this thesis has had for me just as much value as any thesis whatever of the ordinary 'propositional calculus'[41].

The system which he constructed at this time, system $\mathfrak{S}_1$, differs from system $\mathfrak{S}$ in two important respects:

(1) It contains no free variables.

(2) It allows variables to be introduced in the semantic category of any constant that can be defined in the system.

Leśniewski's investigations of possible axioms showed that the law of extensionality in its purely equivalential form

$$[pq] : p \equiv q . \equiv : [f] : f(p) . \equiv . f(q)\big)$$

is too weak to serve as the only axiom to be added to those of system $\mathfrak{S}$, since we cannot prove on such a basis, for example, the law of bivalence in the form

$$[gp] : g(p) . g(\sim p) . \equiv . [q] . g(q)$$

while given some thesis guaranteeing bivalence we can prove the law of extensionality for sentences[42]. He therefore chose the following axiom system for system $\mathfrak{S}_1$:

Ax. I $[pqr] \therefore p \equiv r . \equiv : q \equiv p . \equiv . r \equiv q$
Ax. II $[pqr] \therefore p . \equiv . q \equiv r : \equiv : p \equiv q . \equiv . r$
Ax. III $[gp] :: [f] :: g(p,p) . \equiv \therefore [r] : f(r,r) . \equiv . g(p,p) : \equiv : [r] : f(r,r) . \equiv . g(p . \equiv . [q] . q, p) :: \equiv . [q] . g(q,p)$

The first two axioms correspond to the axioms of system $\mathfrak{S}$, while *Ax. III* is a version of the law of bivalence, stated using equivalence to express conjunction in manner similar to Tarski's first, longer definition. At this time Leśniewski knew that *Ax. III* was equivalent in the context of system $\mathfrak{S}_1$ to the shorter thesis

$$[gp] :: [f] :: g(p) . \equiv \therefore [r] : f(r) . \equiv . g(p) : \equiv : [r] : f(r) . \equiv . g(p . \equiv . [q] . q) :: \equiv . [q] . g(q)$$

but he preferred the longer thesis as an axiom because the terms in it belong to just two semantic categories, those with indices 's' and '$\frac{\mathrm{s}}{\mathrm{s\ s}}$', while the shorter thesis also contains

[40] Leśniewski29, p. 30.
[41] Ibid.
[42] Leśniewski29, p. 43.

terms belonging to the category with the index '$\frac{\mathrm{S}}{\mathrm{S}}$'. He compared this restriction to the exercise of reducing the number of primitive terms in an axiom system[43].

In the authentic symbolism of protothetic, the three axioms given above correspond to the following three theses:

$$A1 \quad \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big) \urcorner$$

$$A2 \quad \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(pq)r\big)\Big) \urcorner$$

$$A3 \quad \llcorner gp \lrcorner \ulcorner \phi\Bigg(\llcorner f \lrcorner \ulcorner \phi\bigg(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g(pp)\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g\big(\phi(p\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big) \urcorner\Big)\bigg) \urcorner \llcorner q \lrcorner \ulcorner g(qp) \urcorner\Bigg) \urcorner$$

These same axioms serve as the basis not only of $\mathfrak{S}_1$ but also of $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{S}_5$.

The directives of system $\mathfrak{S}_1$ can be described as follows:

(α) The directive for detachment of equivalences. Roughly speaking, this allows us to infer from expressions of types '$\phi(\alpha\beta)$' and 'α' the corresponding expression of type 'β'. The directive does not permit detachment 'under' a quantifier; that is, from '$\llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pp)\phi\big(\phi(pq)\phi(qp)\big)\Big) \urcorner$' and '$\llcorner p \lrcorner \ulcorner \phi(pp) \urcorner$' we cannot directly infer '$\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner$'[44].

(β) The directive for substitution. This permits us to substitute expressions for the variables bound by the main quantifier of a thesis. For a given variable we may substitute a variable, a constant, a function, or a generalisation, provided that the result does not violate the mechanism for preserving semantic categories or break some other restriction. 'Fregean' substitution of some expression for an entire function '$f(xy)$' is not permitted[45].

(γ) The directive for distributing universal quantifiers over an equivalence. This directive is described at greater length below.

(δ) The directive for writing definitions having the form of an equivalence which may, if necessary, stand 'under' a universal quantifier binding its variables. The *definiendum* appears as the first argument of the equivalence, and the *definiens* appears as the second argument. The restrictions placed on definitions make the terminological explanation for this directive the most complex of all those published for systems of protothetic[46].

(ζ) A further directive concerning quantifiers. We know very little about this directive. Leśniewski intended that it should allow us to prove after a certain point in the development of $\mathfrak{S}_1$ the equivalence of an expression of the type '$p.\supset.[xy].f(xy)$' with the related expression of the type $[xy]{:}p.\supset.f(xy)$, where 'x' and 'y' may be

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[43] Leśniewski29, pp. 32–3. In his remarks Leśniewski did not use Ajdukiewicz's index notation, which had not yet been invented when Leśniewski29 was published.

[44] Leśniewski29, p. 34.

[45] Leśniewski29, p. 77. Cf. Frege93, p. 63.

[46] In Leśniewski29, pp. 34–5, this directive is described under two headings, δ and ϵ. I follow Sobociński60 in using δ and ϵ respectively for the definition and extensionality directives.

one or more variables belonging to any arbitrary semantic categories, '$f(xy)$' is an expression in which those variables are free, and 'p' is an expression in which none of those variables are free[47]. We know that in its formulation the directive was not allowed to refer to defined terms such as the implication sign, and that in consequence the explanation of this directive was extremely complex[48].

The directive γ for the distribution of quantifiers, when applied to the axiom *A1*, allows us to add any of the following expressions to the system as a new thesis:

$$\phi\Big(\llcorner pqr\lrcorner\ulcorner\phi\big(\phi(pr)\phi(qp)\big)\urcorner\llcorner qr\lrcorner\ulcorner\phi(rq)\urcorner\Big)$$
$$\llcorner r\lrcorner\ulcorner\phi\Big(\llcorner pq\lrcorner\ulcorner\phi\big(\phi(pr)\phi(qp)\big)\urcorner\llcorner q\lrcorner\ulcorner\phi(rq)\urcorner\Big)\urcorner$$
$$\llcorner qpr\lrcorner\ulcorner\phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big)\urcorner$$
$$\llcorner pqr\lrcorner\ulcorner\phi\Big(\phi\big(\phi(pr)\phi(qp)\big)\phi(rq)\Big)\urcorner$$

Note that (1) some, all, or none of the terms may be distributed; (2) the directive does not specify what order the terms shall have in the main quantifier or in the quantifiers of the two arguments of the equivalence; (3) a term cannot move from the main quantifier into an argument's quantifier unless it binds some variable in that argument; (4) a term distributed to one argument may not remain in the main quantifier; (5) the main quantifier must be distributed completely before detachment can take place.

3.5. The first complete system of protothetic

There are a number of meaningful expressions which cannot be proved or disproved on the basis of the axioms and directives of system $\mathfrak{S}_1$. Leśniewski was particularly concerned about two kinds of expression[49]:

(*a*) Expressions which determine the extensionality of all expressions in the semantic category of a particular functor, such as

$$[fg]\therefore[pq]:f(p,q).\equiv.g(p,q):\equiv:[\phi]:\phi\underset{1}{<}f\underset{1}{>}.\equiv.\phi\underset{1}{<}g\underset{1}{>}$$

(*b*) Expressions which Tarski described as 'theorems on the bounds of a function'[50], one of which is, for example,

$$[\phi]:\phi<vr>.\phi<as>.\phi<\sim>.\phi<fl>.\equiv.[f].\phi<f>$$

in which the constants 'vr', 'as', '$\sim$', and 'fl' correspond respectively to the basic constants '$\vdash\!\dashv$', '$\dashv$', '$\vdash$', and '$-$'.

Leśniewski could easily prove expressions of type a if he had available the appropriate expressions of type b, but he suspected that, by analogy with the case of propositional extensionality and bivalence, the converse was not the case; that is, he did not believe that

[47] Cf. Leśniewski29, p. 38.

[48] Ibid.

[49] Leśniewski29, pp. 42–3.

[50] E. g., Tarski56, p. 21.

he could prove an expression of type b even if he had available any number of expressions of type a[51]. He therefore considered solving the problem of the undecidability of certain expressions in $\mathfrak{S}_1$ by adding expressions of type b to the system. Since there is no limit to the number of such expressions, and since they appear to be independent of each other, it is not possible to make system $\mathfrak{S}_1$ complete by adding a finite number of axioms equivalent to expressions of type b. But Leśniewski wanted a complete system, and one in which, if possible, all of the above expressions are provable.

A solution was suggested by a rule which Łukasiewicz had published two years previously:

> I assert each expression containing variables in universal quantifiers from which, through setting the values 0 and 1 in place of the variables, there arise nothing but asserted expressions[52].

After reflecting on this rule, Leśniewski added a new directive to the system:

> In the year 1922 I made the system $\mathfrak{S}_1$ complete by means of a new directive η, which is modelled after the pattern of the above cited directive d of Łukasiewicz, and which in general concerns all variables appearing in the system $\mathfrak{S}_1$ which are not sentence variables. The directive η permitted me to add to the system a new thesis T beginning with a universal quantifier, if there already belonged to the system the theses which you could get from thesis T if you substituted in it for the variables mentioned certain constant function signs whose method of definition is determined in advance with complete precision for all 'semantic categories'[53].

Speaking very informally, Leśniewski said of the resulting system $\mathfrak{S}_2$ that

> System $\mathfrak{S}_2$ is an absolutely 'finite' system, since it permits us to establish for the variables of each 'semantic category' appearing in the system a precisely determined finite number of different possible values: two values for sentence variables (the 'zero' and 'one' of the traditional 'propositional calculus'), four values for variable function signs of sentence-forming functors with one sentence argument, sixteen values for variable function signs of sentence-forming functions which belong to the 'semantic category' to which the function signs of sentence-forming functions of one sentence argument belong, and so on. These possible values of the variables of every given 'semantic category' correspond to the above mentioned constant function signs to which the directive η refers[54].

After mentioning that the terminological explanations required for formalising this directive were extremely complicated, Leśniewski promised to explain it in greater detail in a future instalment of *Grundzüge* which was never in fact published[55]. The question is of considerable interest from several points of view. A number of techniques which are used in deductions in the system $\mathfrak{S}_5$ are not valid when the definition directive is restricted as above, to limit the number of values in each category. It would be a simple matter to develop alternative techniques if the directive η applied to sentence variables, but this is explicitly denied in the first passage quoted. The directive is of considerable interest to students of

[51] Leśniewski29, p. 43.

[52] Łukasiewicz20, p. 197.

[53] Leśniewski29, p. 36.

[54] Leśniewski29, p. 37.

[55] Ibid.

computative protothetic, since it is the direct ancestor of the directives h in those systems. How can a definition directive be formulated in such a manner that the definitions of all constants are specified in advance with complete precision [*vollständig genau im voraus bestimmt*][56]?

3.6. The second complete system of protothetic

Leśniewski clearly saw two of the directives of $\mathfrak{S}_2$ as blots on his system: the quantifier directive ζ and the verification directive η. With the help of his doctoral student Alfred Tarski he spent a good deal of time revising and simplifying them. In both cases Tarski contributed significantly to the ultimate resolution of the difficulty.

In 1922 Tarski showed that, without appealing to the directive ζ, we can prove in systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ all theses of the types

$$[f]\therefore 0.\supset.[xy].f(x,y):\equiv:[xy]:0.\supset.f(x,y)$$
$$[f]\therefore 1.\supset.[xy].f(x,y):\equiv:[xy]:1.\supset.f(x,y)$$

From these two theses we can use the verification procedure for sentence variables to obtain

$$[fp]\therefore p.\supset.[xy].f(x,y):\equiv:[xy]:p.\supset.f(x,y)$$

This means that in systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ the directive ζ is superfluous and can be dropped[57].

In the same year Tarski sketched a general method of proving any 'theorem on the bounds of a function' from the corresponding 'extensionality thesis', provided a similar 'theorem on the bounds' has already been proved for expressions in the same semantic category as each argument of the function in question. This discovery made it possible to replace directive η by a new and much simpler directive, which we shall call ϵ. This directive permits us to add to the system an extensionality thesis for functors in any semantic category. System $\mathfrak{S}_3$ is what Leśniewski called a system of protothetic based on axioms *A1*, *A2*, and *A3* and on directives α, β, γ, δ, and ϵ[58]. In this system Leśniewski returned to the more liberal definition directive which allows us to define any number of constants in any semantic category.

In the following year Tarski was studying the problem of reducing the number of axioms in various deductive systems. In the course of his research he determined that, given a thesis which states that $\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner$, we can use the directives of $\mathfrak{S}_3$ to derive from some thesis of the type $\llcorner fp \lrcorner \ulcorner \phi\Big(f(Pp)\phi\big(f(Qp)P\big)\Big) \urcorner$, which is a kind of 'logical product', the corresponding expressions P and Q. We do this by defining a function '$\text{-}\phi\text{-}$' for which we can prove that $\llcorner pq \lrcorner \ulcorner \text{-}\phi\text{-}(pq) \urcorner$, then detaching this twice from the 'logical product'. From this he inferred that system $\mathfrak{S}_3$ can be based on two axioms, one of which is a thesis equivalent to $\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner$, and the other of which is a 'logical product' of any arbitrary number of theses[59].

[56] Leśniewski29, p. 36.

[57] Leśniewski29, pp. 38–41.

[58] Leśniewski29, p. 44, where the extensionality directive is called '$\eta*$'. I use 'ϵ' following Sobociński60, p. 57.

[59] Leśniewski29, pp. 50–4.

3.7. The ‘official’ system of protothetic

In 1923 Leśniewski observed that if he modified the definition directive δ to require that the new term should appear in the second argument of the equivalence while the *definiens* is the first argument, then from a ‘logical product’ of the type just mentioned we can infer P without appealing to a thesis of the type $\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner$[60]. Moreover, if P is in fact a thesis of this type, we can infer the other conjuncts of the ‘logical product’, and we could have a system of protothetic which is based on a single axiom. The ‘official’ system of protothetic $\mathfrak{S}_5$ is based on the axioms *A1*, *A2*, and *A3* and the directives α, β, γ, δ with the *definiendum* on the right, and ϵ. Leśniewski published more than eighty pages of deductions in system $\mathfrak{S}_5$, including proofs of theses corresponding to Łukasiewicz’s axiom system for the ‘propositional calculus’ based on implication and negation[61] and a form of the law of extensionality for sentences[62]:

$$T422 \quad \llcorner pqs \lrcorner \ulcorner \phi\!\!-\Big(\phi\!\!-(sq)\phi\!\!-\big(\phi\!\!-(qp)\phi\!\!-(sp)\big)\Big) \urcorner$$

$$T400 \quad \llcorner pq \lrcorner \ulcorner \phi\!\!-\Big(p\phi\!\!-\big(\vdash(p)q\big)\Big) \urcorner$$

$$T398 \quad \llcorner p \lrcorner \ulcorner \phi\!\!-\Big(\phi\!\!-\big(\vdash(p)p\big)p\Big) \urcorner$$

$$T381 \quad \llcorner gpq \lrcorner \ulcorner \phi\!\!-\Big(\phi(pq)\phi\big(g(p)g(q)\big)\Big) \urcorner$$

The first single axiom of a system equivalent to $\mathfrak{S}_5$ was 290 words long[63]:

$$\begin{aligned} A_a \quad & \llcorner fp \lrcorner \ulcorner \phi\Bigg(f\Big(\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner p\Big)\phi\Bigg(f\Bigg(\llcorner hs \lrcorner \ulcorner \phi\Bigg(h\Big(\llcorner pqr \lrcorner \ulcorner \phi\big(\phi(\phi(pr)\phi(qp))\phi(rq)\big) \urcorner s\Big)\phi\Bigg(\\ & h\Bigg(\llcorner kt \lrcorner \ulcorner \phi\Bigg(k\Big(\llcorner pqr \lrcorner \ulcorner \phi\big(\phi(p\phi(qr))\phi(\phi(pq)r)\big) \urcorner t\Big)\phi\Bigg(k\Bigg(\llcorner gp \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \ulcorner \phi\Big(g(pp)\phi\Big(\llcorner r \lrcorner \ulcorner \phi\big(f(rr)g \\ & (pp)\big) \urcorner \llcorner r \lrcorner \ulcorner \phi\Big(f(rr)g\big(\phi(p\llcorner q \lrcorner \ulcorner q \urcorner)p\big)\Big) \urcorner \Big)\Big) \urcorner \llcorner q \lrcorner \ulcorner g(qp) \urcorner \Bigg) \urcorner t\Bigg) \llcorner pqr \lrcorner \ulcorner \phi\big(\phi(p\phi(qr))\phi(\phi(pq)r)\big) \urcorner \Bigg)\Bigg) \\ & \urcorner s\Bigg) \llcorner pqr \lrcorner \ulcorner \phi\big(\phi(\phi(pr)\phi(qp))\phi(rq)\big) \urcorner \Bigg)\Bigg) \urcorner p\Bigg) \llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner \Bigg)\Bigg) \urcorner \end{aligned}$$

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60 Leśniewski29, pp. 54–5.

61 Łukasiewicz29, pp. 45 and 66–98.

62 Leśniewski39, pp. 137, 139, 143–4.

63 Leśniewski38, pp. 24–5. Cf. Sobociński60, p. 64.

Several shorter axioms found in 1923 are not significantly different from A_a, since they merely involve slight changes in the various conjuncts forming the basis of system $\mathfrak{S}$; in all of them one conjunct is essentially identical with axiom *A3*. The last single axiom of this general type was discovered by Leśniewski in 1926; it is 124 words long and is essentially the 'logical product' of Leśniewski's axiom *A3* and Wajsberg's axiom *W4*[64]:

$$A_d \quad \llcorner fhpqrx \lrcorner \ulcorner \phi\Big(f\Big(\llcorner k \lrcorner \ulcorner \phi\Big(\llcorner s \lrcorner \ulcorner \phi(k(ss)h(pp))\urcorner \phi\Big(h(pp)\llcorner s \lrcorner \ulcorner \phi\Big(k(ss)h\big(\phi(p\llcorner t \lrcorner \ulcorner t \urcorner)p\big)\Big)\urcorner\Big)\Big)\urcorner q\Big)\phi\Big(f\big(\llcorner t \lrcorner \ulcorner h(tp)\urcorner q\big)\phi\Big(\phi\Big(\phi\big(p\phi(qr)\big)\phi\big(\phi(rx)x\big)\Big)\phi(pq)\Big)\Big)\Big)\urcorner$$

In 1926 Leśniewski suggested to Wajsberg that this axiom might be shortened if instead of *W4* it contained some thesis from which we could prove not only the theses of system $\mathfrak{S}$ but also a thesis of extensionality, since this would allow the form of the 'logical product' in the axiom to be simplified. Shortly afterwards Wajsberg discovered that all theses of system $\mathfrak{S}$ can be proved from the axiom *W5*, which, though it is not strictly speaking an extensionality thesis, nevertheless allows us to simplify the form of the 'logical product' in an axiom:

$$W5 \quad \llcorner pqrst \lrcorner \ulcorner \phi\Big(\phi(pq)\llcorner g \lrcorner \ulcorner \phi\Big(g\Big(\phi\big(\phi(rs)t\big)q\Big)g\Big(\phi\big(\phi(st)r\big)p\Big)\Big)\urcorner\Big)\urcorner$$

Note that after defining $\llcorner p \lrcorner \ulcorner \phi\big(p\dashv(p)\big)\urcorner$ we can obtain in two detachments from *W5* the thesis[65]

$$\llcorner rst \lrcorner \ulcorner \phi\Big(\phi\big(\phi(st)r\big)\phi\big(\phi(rs)t\big)\Big)\urcorner$$

Both this type of detachment and various relatives of the law of extensionality have played an important rôle in almost all subsequent single axioms. Making use of these discoveries Wajsberg constructed in 1926 an axiom 120 words long[66]:

$$A_e \quad \llcorner fhpqrst \lrcorner \ulcorner \phi\Big(f\Big(\llcorner g \lrcorner \ulcorner \phi\Big(h(pp)\phi\Big(g\big(qh(pp)\big)g\Big(qh\big(\phi(p\llcorner t \lrcorner \ulcorner t \urcorner)p\big)\Big)\Big)\Big)\urcorner r\Big)\phi\Big(f\big(\llcorner t \lrcorner \ulcorner h(tp)\urcorner r\big)\phi\Big(\phi(pq)\llcorner g \lrcorner \ulcorner \phi\Big(g\Big(\phi\big(\phi(rs)t\big)q\Big)g\Big(\phi\big(\phi(st)r\big)p\Big)\Big)\urcorner\Big)\Big)\Big)\urcorner$$

In the same year 1926 Leśniewski discovered the axiom A_f, which is 116 words long. It is essentially a 'logical product' of a simplification of *A3* with Wajsberg's axiom *W3*[67]:

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64 Leśniewski38, p. 27.
65 Leśniewski38, p. 32.
66 Leśniewski38, p. 29.
67 Leśniewski38, p. 30.

$$A_f \quad \lfloor fhpqrs \rfloor \ulcorner \varphi \Bigg(f\big(\lfloor t \rfloor \ulcorner h(tp) \urcorner q\big) \varphi \Bigg(f\Bigg(\lfloor k \rfloor \ulcorner \varphi \Big(h(pp) \varphi \Big(\lfloor s \rfloor \ulcorner k(ss) \urcorner \lfloor s \rfloor \ulcorner \varphi \Big(k(ss) h\big(\varphi(p \lfloor t \rfloor \ulcorner t \urcorner) p\big)\Big) \urcorner$$

$$\Big)\Big) \urcorner q \Bigg) \varphi \Big(\varphi \Big(\varphi \big(\varphi (\varphi(pq) r\big) s \Big) \varphi \Big(s \varphi \big(p \varphi (qr) \big)\Big)\Big)\Bigg)\Bigg) \urcorner$$

Later in 1926 Wajsberg discovered the axiom A_g, which is 106 words long. This axiom, which is based on the axioms *W2a* and *W2b*, makes use of a number of interesting *ad hoc* devices, though the initial deductions from it are not as obvious as those starting with earlier axioms[68]:

$$A_g \quad \lfloor fp \rfloor \ulcorner \varphi \Bigg(\lfloor s \rfloor \ulcorner f(sp) \urcorner \lfloor g \rfloor \ulcorner \varphi \Bigg(f(pp) \varphi \Bigg(\lfloor t \rfloor \ulcorner g(tt) \urcorner \lfloor qrt \rfloor \ulcorner \varphi \Big(g \Big(\varphi \big(\varphi ((tt)t\big) t \Big) \varphi \Big(f\big(\varphi(p \lfloor s \rfloor \ulcorner s \urcorner) p\big) \varphi \Big(\varphi \Big($$

$$p \varphi(qr) \big) \varphi \big(r \varphi(qp) \big)\Big)\Big)\Big) \urcorner \Bigg)\Bigg) \urcorner \Bigg) \urcorner$$

Leśniewski was able to produce a much shorter axiom by departing completely from the style of axiom which resembles *A3*. All known single axioms which are shorter than A_g are based in some respect on the following thesis discovered in about 1922 by Tarski[69]:

$$\lfloor f \rfloor \ulcorner \varphi \Big(f\big(f(\lfloor u \rfloor \ulcorner u \urcorner)\big) \lfloor p \rfloor \ulcorner f(p) \urcorner \Big) \urcorner$$

This thesis, together with the law of extensionality, permits us to dispense with the ‘logical product’ stating, in effect, that

$$\lfloor fp \rfloor \ulcorner \varphi \Big(\varphi \Big(f(p) f\big(\varphi(p \lfloor u \rfloor \ulcorner u \urcorner)\big)\Big) \lfloor q \rfloor \ulcorner f(q) \urcorner \Big) \urcorner$$

Leśniewski discovered that he could replace such a thesis with one more or less like the following thesis, which incorporates characteristics both of the law of extensionality and of the thesis discovered by Tarski:

$$A3a \quad \lfloor pq \rfloor \ulcorner \varphi \Big(\varphi(pq) \lfloor f \rfloor \ulcorner \varphi \Big(f\big(pf(p \lfloor u \rfloor \ulcorner u \urcorner)\big) \lfloor r \rfloor \ulcorner f(qr) \urcorner \Big) \urcorner \Big) \urcorner$$

If we have obtained two theses of the type ‘$f(\lfloor u \rfloor \ulcorner u \urcorner)$’ and ‘$f(1)$’, where ‘$f$’ is any expression whatever containing the ‘argument’ or ‘arguments’, and ‘1’ is any expression which has been proved in the system, thesis *A3a* allows us to prove, with the help of certain theses of system $\mathfrak{S}$, the corresponding thesis of the type ‘$\lfloor r \rfloor \ulcorner f(r) \urcorner$’. Thus *A3a* effectively replaces the thesis *A3*. This procedure (or any procedure which has an equivalent effect) is so important in most deductions in system $\mathfrak{S}_5$ that it is useful to have a name for it: we shall call it *M5*[70]. This derived ‘metarule’ can be established using *A3a* if we introduce a functor ‘ϕ’ corresponding to the expression ‘f’ with the definition

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[68] Leśniewski38, p. 30.

[69] Cf. Tarski56, p. 17.

[70] Cf. Sobociński61, pp. 119–20, where the metarule is called *S5*.

$$\llcorner pq \lrcorner \ulcorner \varphi\Big(\varphi\big(f(p)\varphi(qq)\big)\phi(qp)\Big) \urcorner$$

System $\mathfrak{S}$ allows us to prove on this basis that $\llcorner p \lrcorner \ulcorner \varphi\big(1\phi(p \llcorner u \lrcorner \ulcorner u \urcorner)\big) \urcorner$ and that $\llcorner p \lrcorner \ulcorner \phi(p1) \urcorner$. The ‘extensional’ ability of *A3a* allows us to prove from these expressions that $\llcorner p \lrcorner \ulcorner \phi\big(p\phi(p \llcorner u \lrcorner \ulcorner u \urcorner)\big) \urcorner$. We can then use the part of *A3a* which corresponds to Tarski’s thesis to prove that $\llcorner pr \lrcorner \ulcorner \phi(pr) \urcorner$. From this and the above definition of ‘ϕ’ we can prove, with the help of system $\mathfrak{S}$, that $\llcorner r \lrcorner \ulcorner f(r) \urcorner$.

By relying on deductions like these, Leśniewski constructed in 1926 a single axiom of protothetic only 82 words in length[71]:

$$A_h \quad \llcorner fpqrst \lrcorner \ulcorner \varphi\Bigg(\varphi(pq) \llcorner g \lrcorner \ulcorner \varphi\Bigg(f\big(pf(p \llcorner u \lrcorner \ulcorner u \urcorner)\big)\varphi\Bigg(\llcorner u \lrcorner \ulcorner f(qu) \urcorner \varphi\Big(g\Big(\varphi\big(\varphi(rs)t\big)q\Big)g\Big(\varphi\big(\varphi(st)r\big)p\Big)\Big)\Bigg)\Bigg) \urcorner \Bigg) \urcorner$$

Thesis A_h is an *organic*[72] amalgamation of thesis *A3a* and Wajsberg’s thesis *W5*. It does not seem to have been observed that we can distribute all terms from the main quantifier except p and q and obtain an axiom equivalent to A_h but more ‘canonical’[73]. Moreover this thesis has the defect that it effectively duplicates the expression of ‘extensionality’, since both functor variables ‘f’ and ‘g’ are used for this purpose. Leśniewski would have preferred an axiom from which it is easier to derive one of the ordinary axiom systems for $\mathfrak{S}$[74]. He discovered such a thesis in 1933 when he learned of Łukasiewicz’s discovery of the single axioms *W12*, *W13*, and *W14*; like A_h it is 82 words long[75]:

$$A_i \quad \llcorner fpqrst \lrcorner \ulcorner \varphi\Bigg(\varphi(pq) \llcorner g \lrcorner \ulcorner \varphi\Bigg(f\big(pf(p \llcorner u \lrcorner \ulcorner u \urcorner)\big)\varphi\Bigg(\llcorner u \lrcorner \ulcorner f(qu) \urcorner \varphi\Big(g\Big(\varphi\big(\varphi(rt)\varphi(sr)\big)q\Big)g\big(\varphi(st)p\big)\Big)\Bigg)\Bigg) \urcorner \Bigg) \urcorner$$

We use Leśniewski and Tarski’s procedure described in section 3.6 to define a functor which is true for all arguments, and with this and Wajsberg’s procedure (as used in the initial deductions from *W5*), we detach from A_i the thesis

$$\llcorner rst \lrcorner \ulcorner \varphi\Big(\varphi(st)\varphi\big(\varphi(rt)\varphi(sr)\big)\Big) \urcorner$$

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71 Leśniewski29, p. 59; Leśniewski38, pp. 30–1

72 The term ‘organic’ refers to a thesis no part of which is a thesis or becomes a thesis when bound by an appropriate quantifier. Leśniewski introduced the concept of organic theses, and Mordchaj Wajsberg gave the first formal definition of the concept. See Sobociński56, p. 60, and ŁukasiewiczTarski30, p. 37.

73 See Sobociński56, p. 62, for a definition and discussion of this term.

74 Leśniewski38, p. 31.

75 Leśniewski38, p. 32.

which corresponds to Łukasiewicz's axiom *W12* for system $\mathfrak{S}$. Leśniewski outlines at length the remaining deductions from A_i, establishing on that basis the metarule we have called *M5* and sketching the proof of theses expressing the 'laws of conjunction'[76]. At the end of his outline, he says that, in the light of his remarks 'we can "see" that axiom A_i is sufficient for constructing' a system of protothetic[77]. Sobociński later revealed that Leśniewski's remark refers to the following important metatheorem proved by Leśniewski, which we shall call *L1*[78]:

L1 An axiom system is adequate as a basis for a complete system of protothetic having the directives of $\mathfrak{S}_5$ if from the system we can prove

(1) One or more theses adequate for proving all theses of system $\mathfrak{S}$.

(2) The metarule *M5*.

(3) The four laws of conjunction:

$$K1 \quad \phi\big(\llcorner u\lrcorner\ulcorner u\urcorner \text{ϙ}(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\big)$$
$$K2 \quad \phi\Big(\llcorner u\lrcorner\ulcorner u\urcorner \text{ϙ}\big(\llcorner u\lrcorner\ulcorner u\urcorner\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\big)\Big)$$
$$K3 \quad \phi\Big(\llcorner u\lrcorner\ulcorner u\urcorner \text{ϙ}\big(\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\llcorner u\lrcorner\ulcorner u\urcorner\big)\Big)$$
$$K4 \quad \text{ϙ}\big(\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)\big)$$

In 1937 Sobociński observed that in metatheorem *L1* condition 3 is equivalent to requiring that the following two theses be provable[79]:

$$S1 \quad \llcorner pq\lrcorner\ulcorner\phi\Big(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\big(f(p)f(q)\big)\urcorner\Big)\urcorner$$
$$S2 \quad \llcorner pq\lrcorner\ulcorner\phi\Bigg(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(p)f(q)\big)\phi(pq)\Big)\urcorner\Bigg)\urcorner$$

In the same year he proved that, given conditions 1 and 2 of metatheorem *L1*, the theses *S1* and *S2* are deductively equivalent. This allowed him to construct an axiom 72 words long[80]:

$$A_j \quad \llcorner pq\lrcorner\ulcorner\phi\Bigg(\phi(pq)\llcorner fst\lrcorner\ulcorner\phi\Bigg(f\big(pf(p\llcorner u\lrcorner\ulcorner u\urcorner)\big)\phi\Bigg(\llcorner r\lrcorner\ulcorner f(qr)\urcorner\phi\Big(\phi\big(\phi(st)q\big)\phi\big(t\phi(sp)\big)\Big)\Bigg)\Bigg)\urcorner\Bigg)\urcorner$$

When Leśniewski learned of axiom A_j he was able almost immediately to construct an equivalent axiom 71 words long[81]:

$$A_k \quad \llcorner pq\lrcorner\ulcorner\phi\Bigg(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Bigg(f\big(pf(p\llcorner u\lrcorner\ulcorner u\urcorner)\big)\llcorner rs\lrcorner\ulcorner\phi\Big(f(qr)\phi\Big(\phi\big(\phi(sr)q\big)\phi\big(r\phi(sp)\big)\Big)\Big)\urcorner\Bigg)\urcorner\Bigg)\urcorner$$

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[76] Leśniewski38, pp. 31–5.
[77] Leśniewski38, p. 35.
[78] Sobociński49, pp. 16–7.
[79] Sobociński54, p. 19.
[80] Ibid.
[81] Sobociński60, p. 66. In Sobociński54, p. 19, there is a slightly different axiom with this name, also 71 words long, but not organic.

In 1938 Sobociński was able, by using a theses added to the system in accordance with the extensionality directive ϵ, to prove thesis *S1* from conditions 1 and 2 of metarule *L1*. This means that condition 3 of metarule *L1* is superfluous. He used this information to establish that the following thesis, 66 words in length, is adequate as a single axiom of protothetic[82]:

$$A_l \quad \llcorner fpq\lrcorner \ulcorner\phi\Bigl(f\bigl(pf(p\llcorner u\lrcorner\ulcorner u\urcorner)\bigr)\llcorner r\lrcorner\ulcorner\phi\Bigl(f\Bigl(\phi\bigl(\phi(pq)q\bigr)r\Bigr)\phi\Bigl(\phi(pq)\phi\bigl(\phi(rq)\phi(pr)\bigr)\Bigr)\Bigr)\urcorner\Bigr)\urcorner$$

Sobociński suggested to Leśniewski that condition 1 of metatheorem *L1* could probably be weakened. In fact axioms A_j and A_k had already taken advantage of this, in a sense. Reflecting on this and on the work which led to the discovery of axiom A_l, Leśniewski established in 1938 that a 62-word thesis could serve as the sole axiom of a system of protothetic[83]:

$$A_m \quad \llcorner pq\lrcorner \ulcorner\phi\Bigl(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Bigl(f\bigl(qf(q\llcorner u\lrcorner\ulcorner u\urcorner)\bigr)\llcorner r\lrcorner\ulcorner\phi\Bigl(f(pr)\phi\bigl(r\phi\bigl(q\phi(rp)\bigr)\bigr)\Bigr)\urcorner\Bigr)\urcorner\Bigr)\urcorner$$

In 1945 Sobociński discovered four axioms 54 words in length, and thus shorter than A_m[84]:

$$A_n \quad \llcorner pq\lrcorner \ulcorner\phi\Bigl(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Bigl(f\bigl(pf(p\llcorner u\lrcorner\ulcorner u\urcorner)\bigr)\llcorner r\lrcorner\ulcorner\phi\bigl(f(qr)\phi(qp)\bigr)\urcorner\Bigr)\urcorner\Bigr)\urcorner$$

$$A_o \quad \llcorner pq\lrcorner \ulcorner\phi\Bigl(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Bigl(f\bigl(qf(q\llcorner u\lrcorner\ulcorner u\urcorner)\bigr)\llcorner r\lrcorner\ulcorner\phi\bigl(f(pr)\phi(qp)\bigr)\urcorner\Bigr)\urcorner\Bigr)\urcorner$$

$$A_p \quad \llcorner pq\lrcorner \ulcorner\phi\Bigl(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Bigl(f\bigl(pf(q\llcorner u\lrcorner\ulcorner u\urcorner)\bigr)\llcorner r\lrcorner\ulcorner\phi\bigl(f(qr)\phi(qp)\bigr)\urcorner\Bigr)\urcorner\Bigr)\urcorner$$

$$A_q \quad \llcorner pq\lrcorner \ulcorner\phi\Bigl(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\Bigl(f\bigl(qf(p\llcorner u\lrcorner\ulcorner u\urcorner)\bigr)\llcorner r\lrcorner\ulcorner\phi\bigl(f(pr)\phi(qp)\bigr)\urcorner\Bigr)\urcorner\Bigr)\urcorner$$

In 1952, reflecting on metatheorem *L1*, Sobociński tried to formulate an equivalent metatheorem in which the conditions are the simplest possible. After several simplifications he was able to prove the following metatheorem[85]:

L2 An axiom system is adequate as a basis for a complete system of protothetic having the directives of $\mathfrak{S}_5$ if from the system we can prove

(1) The following two theses:

F1 $\phi(\llcorner u\lrcorner\ulcorner u\urcorner\llcorner u\lrcorner\ulcorner u\urcorner)$

F2 $\llcorner pq\lrcorner\ulcorner\phi\bigl(p\phi\bigl(\phi(qp)q\bigr)\bigr)\urcorner$

(2) The metarule *M5*.

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[82] Sobociński49, pp. 17, 26.

[83] Sobociński49, pp. 18, 27.

[84] Sobociński49, pp. 18, 27; Sobociński60, p. 67.

[85] Sobociński54, p. 19.

The possibility of defining either of the terms for implication and conjunction by means of the other, for example, by one of the definitions

$$D1 \quad \llcorner pq \lrcorner \ulcorner \phi\Big(\phi\big(p\phi\!\text{-}(pq)\big)\text{ϙ}(pq)\Big) \urcorner$$

$$D2 \quad \llcorner pq \lrcorner \ulcorner \phi\Big(\phi\big(p\text{ϙ}(pq)\big)\phi\!\text{-}(pq)\Big) \urcorner$$

shows that in metatheorem *L1* we can easily replace condition 3 by the equivalent condition that the four laws of implication must be provable. This equivalent condition is more directly useful in deductions starting from the conditions of metatheorem *L1*, and often it is in fact easier to prove the four laws of implication directly. Between 1975 and 1979 I studied carefully the initial deductions from A_n and from the related axioms. In 1976 I observed that in fact the laws of implication, and hence all conditions of metarule *L1*, can be proved from each of the axioms A_m, A_n, A_o, A_p, and A_q without appealing to the extensionality directive[86]. This means that deductions from these axioms can dispense almost entirely with the theoretical considerations which led to their discovery.

During the same period I showed that there exist further theses having the same length as axiom A_n which can serve as a single axiom of protothetic, and which are trivial variations of those axioms. For example,

$$A_r \quad \llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pq)\llcorner f \lrcorner \ulcorner \phi\Big(f\big(f(\llcorner u \lrcorner \ulcorner u \urcorner p)p\big)\llcorner r \lrcorner \ulcorner \phi\big(f(rq)\phi(qp)\big) \urcorner\Big) \urcorner\Big) \urcorner$$

Although there are at least eight axioms of the same length, this does not suggest that there are shorter axioms, as might be the case in systems of protothetic based on other primitive terms, or in most other deductive theories. In this respect system $\mathfrak{S}_5$ resembles system $\mathfrak{S}$, for which there are at least eleven shortest single axioms, namely theses *W12*–*W22*. It is fairly certain that thesis *A3a*, which is closely related to all of the shortest single axioms of protothetic, is not itself adequate as an axiom of the theory, and that there is no thesis related to *A3a*, shorter than A_n, and adequate as a single axiom for a system of protothetic with directives equivalent to those of $\mathfrak{S}_5$. If there is a shorter single axiom, then its discovery must await some entirely new insight into the deductive structure of the theory.

3.8. Protothetic based on implication

Leśniewski wanted the directives of his systems to be relatively independent of the particular primitive terms on which the systems are based[87]. Obviously there must be some adaptation between primitive terms and directives; if we tried, for example, to use the symbol for implication '$\phi\!\text{-}$' in the axioms of a system whose detachment directive is formalised to apply to the Sheffer functor '$\text{-}\!\text{ტ}\!\text{-}$', we would almost certainly obtain an inconsistent system[88].

In 1922 Leśniewski constructed a system of protothetic based on implication as its primitive term and having directives analogous to those of $\mathfrak{S}_2$, including the verification

[86] LeBlanc85, p. 487.

[87] Leśniewski29, p. 45.

[88] Cf. Sobociński56, p. 56.

directive η[89]. The theoretical considerations which led to the construction of system $\mathfrak{S}_3$ showed that it would be significantly simpler to base the directives of a system of implicational protothetic on the latter system. In 1922 Leśniewski constructed such a system and called it $\mathfrak{S}_4$[90]. The first axiom system of $\mathfrak{S}_4$ and of its unnamed predecessor consisted of the following theses[91]:

$$B1 \quad \llcorner pq \lrcorner \ulcorner \phi\!\!-\!\big(p\,\phi\!\!-\!(qp)\big) \urcorner$$

$$B2 \quad \llcorner pqr \lrcorner \ulcorner \phi\!\!-\!\Big(\phi\!\!-\!(pq)\,\phi\!\!-\!\big(\phi\!\!-\!(qr)\,\phi\!\!-\!(pr)\big)\Big) \urcorner$$

$$B3 \quad \llcorner pqr \lrcorner \ulcorner \phi\!\!-\!\Big(\phi\!\!-\!\big(\phi\!\!-\!(pq)r\big)\,\phi\!\!-\!\big(\phi\!\!-\!(pr)r\big)\Big) \urcorner$$

$$B4 \quad \llcorner gpq \lrcorner \ulcorner \phi\!\!-\!\Bigg(g(pp)\,\phi\!\!-\!\Big(g\big(\phi\!\!-\!(p \llcorner q \lrcorner \ulcorner q \urcorner)p\big)g(qp)\Big)\Bigg) \urcorner$$

in which axiom *B4* is a thesis establishing bivalence in the same way in which axiom *A3* establishes bivalence in the systems based on equivalence.

The directives of system $\mathfrak{S}_4$ can be described as follows:

(α) The directive for detachment of implications. Roughly speaking, this allows us to infer from expressions of types '$\phi\!\!-\!(\alpha\beta)$' and 'α' the corresponding expression of type 'β'. The directive, like the corresponding directive in the equivalential system, does not permit detachment 'under' a quantifier.

(β) The directive for substitution, just like the corresponding directive in the equivalential systems.

(γ) The directive for distributing a universal quantifier over an equivalence, just like the corresponding directive in the equivalential system.

(δ) The directive for writing definitions having the form '$\llcorner r \lrcorner \ulcorner \phi\!\!-\!\Bigg(\phi\!\!-\!\Big(\phi\!\!-\!(PQ)\,\phi\!\!-\!\big(\phi\!\!-\!(QP)r\big)\Big)r\Bigg) \urcorner$', in which apparently '$P$' represents the *definiens* and 'Q' represents the *definiendum*[92].

(ϵ) The directive for writing theses of extensionality. We do not know the precise form employed, although it contained no constant terms other than the primitive symbol for equivalence[93], but it is likely that the theses had a form much like this one:

$$\llcorner fg \lrcorner \ulcorner \phi\!\!-\!\Big(\llcorner p \lrcorner \ulcorner \phi\!\!-\!\big(f(p)g(p)\big) \urcorner \phi\!\!-\!\Big(\llcorner p \lrcorner \ulcorner \phi\!\!-\!\big(g(p)f(p)\big) \urcorner \llcorner \phi \lrcorner \ulcorner \phi\!\!-\!\big(\phi\langle f\rangle \phi\langle g\rangle\big) \urcorner\Big)\Big) \urcorner$$

[89] Leśniewski29, pp. 45–8.

[90] Leśniewski29, p. 48.

[91] Leśniewski29, pp. 47–8.

[92] There is no detailed description of these directives. Leśniewski says in Leśniewski29, p. 46, that he adopted this function in place of an earlier, more complex function, which he does not specify explicitly. 'P' probably represents the *definiens* because of the parallel form for implicational definitions in Leśniewski38, p. 37.

[93] Leśniewski29, p. 48.

In 1922, investigating a suggestion by Leśniewski, Tarski proved that in the system of axioms *B1*, *B2*, *B3*, and *B4*, the last three axioms can be replaced by the following axiom[94]:

$$B5 \quad \llcorner pqrf \lrcorner \ulcorner \phi\!- \Big(f(rp) \phi\!- \Big(f\big(r \phi\!-(p \llcorner s \lrcorner \ulcorner s \urcorner)\big) f(rq) \Big) \Big) \urcorner$$

In 1925 Tarski discovered a general method of constructing single axioms for theories based on implication[95]. If this method is applied to the last mentioned axiom system, we obtain the following single axiom for system $\mathfrak{S}_4$, which contains 70 words:

$$A_s \quad \llcorner pqrs \lrcorner \ulcorner \phi\!- \Bigg(\phi\!- \Bigg(\phi\!- \big(p \phi\!- (qp) \big) \phi\!- \Big(\llcorner pqrf \lrcorner \ulcorner \phi\!- \Big(f(rp) \phi\!- \Big(f\big(r \phi\!-(p \llcorner q \lrcorner \ulcorner q \urcorner)\big) f(rq) \Big) \Big) \urcorner r \Big) \Bigg) \phi\!-(sr) \Bigg) \urcorner$$

This axiom A_s appears to be roughly at the same stage of development as axiom A_h or A_i in terms of the state of investigations of single axioms for system $\mathfrak{S}_5$. That is, we have not yet discovered any shorter axioms; they almost certainly exist, but we shall not discover them until we understand the deductive structure of $\mathfrak{S}_4$ better.

3.9. Computative protothetic

In 1920 Łukasiewicz published an article which attempts 'to interpret two-valued logic in such a way that three-valued logic will prove a natural extension of it'[96]. The resulting system is quite unusual, since the style in which it is constructed does not resemble that of any previous system. As Łukasiewicz had not yet invented his own bracketless notation, he used a Boole-Schröder-Couturat notation with slight modifications. In this system there are signs of assertion and of rejection: the sign of assertion is '$U\!:$' and the sign of rejection is '$N\!:$'. There are three axioms[97]:

T_1	$U\!:\Pi_p\,.\,0 < p$	$([p].0\supset p)$
T_2	$U\!:\Pi_p\,.\,p < 1$	$([p].p\supset 1)$
Z_3	$N\!:1 < 0$	$(1\supset 0)$

In addition to these, the system has a number of definitions[98]:

D_1	$U\!:\Pi_p\,.\,p' \equiv (p < 0)$	$([p]:\sim p.\equiv.p\supset 0)$
D_{2a}	$U\!:\Pi_{pr}\,.\,p+r \equiv [(p<r)<r]$	$([pr]\therefore p\vee r.\equiv:p\supset r.\supset.r)$
D_3	$U\!:\Pi_{pr}\,.\,pr \equiv (p'+r')'$	$([pr]:p.r.\equiv.\sim(\sim p\vee\sim r))$
D_4	$U\!:\Pi_{pr}\,.\,(p=r) \equiv (p<r)(r<p)$	$([pr]\therefore p=r.\equiv:p\supset r.r\supset p)$

The system has four directives[99]:

(a) We can assert any expression which we can obtain from an asserted expression by substituting '0' or '1' for variables bound by the main quantifier of the original asserted expression.

[94] Leśniewski29, p. 50.

[95] Leśniewski29, pp. 58–9.

[96] Łukasiewicz20, p. 189.

[97] Łukasiewicz20, p. 202.

[98] Ibid. Note that '$\equiv$' is used for definition, '$=$' for material equivalence.

[99] Łukasiewicz20, p. 197.

(b) In any asserted or rejected expression we can replace expressions containing a defined term by the corresponding expression not containing a defined term, or *vice versa.* The resulting expression is asserted or rejected according as the original expression is asserted or rejected.

(c) In any asserted or rejected expression we can replace any '1' with any asserted expression, and we can replace any '0' with any rejected expression. The resulting expression is asserted or rejected according as the original expression is asserted or rejected.

(d) We can assert an expression beginning with a universal quantifier if all expressions are asserted which result from the original expression by substituting '0' or '1' for the variables bound by its main quantifier.

Leśniewski had already been influenced by this article in 1922 when he formulated directive η of system $\mathfrak{S}_2$. During the academic year 1924–5 at the University of Warsaw, he proposed this article as a subject for discussion at a seminar he was conducting. In the light of the ensuing discussion, Leśniewski constructed in 1924 a system of protothetic whose style of development resembled the style of Łukasiewicz's system more closely than that of $\mathfrak{S}_5$. Concerning this system he wrote that

> Since I took Łukasiewicz's construction to a very significant extent as a model, I was concerned to employ an 'automatic verification' style in my new system, as opposed to the much more common 'substitution/detachment' style[100].

Leśniewski's new system of 'computative' protothetic was based on the primitive terms 'ϕ-' and 'Λ'. It had as its single axiom the thesis

$$A \quad \phi\text{-}(\Lambda\Lambda)$$

and it is developed with the help of nine directives[101]:

(*a*) Given theses of types 'α' and 'β', we can add the corresponding expression of type '$\phi\text{-}(\alpha\beta)$' as a new thesis.

(*b*) Given theses of types 'α' and '$\phi\text{-}(\beta\Lambda)$', we can add the corresponding expression of type '$\phi\text{-}\big(\phi\text{-}(\alpha\beta)\Lambda\big)$' as a new thesis.

(*c*) Given theses of types '$\phi\text{-}(\alpha\Lambda)$' and '$\beta$', we can add the corresponding expression of type '$\phi\text{-}(\alpha\beta)$' as a new thesis.

(*d*) Given theses of types '$\phi\text{-}(\alpha\Lambda)$' and '$\phi\text{-}(\beta\Lambda)$', we can add the corresponding expression of type '$\phi\text{-}(\alpha\beta)$' as a new thesis.

(*e*) We may add to the system definitions having the form of an expression of the type

$$\phi\text{-}\Bigg(\phi\text{-}\bigg(\phi\text{-}(PQ)\phi\text{-}\Big(\phi\text{-}(QP)\Lambda\Big)\bigg)\Lambda\Bigg)$$

or of the same type enclosed in a univeral quantifier. The expressions 'P' and 'Q' represent respectively the *definiens* and the *definiendum* of the definition.

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[100] Leśniewski38, p. 35.

[101] Leśniewski38, pp. 37–8.

(*f*) Given a definition whose *definiens* is a thesis (or a substitution of whose *definiens* is a thesis) we may add as a new thesis the *definiendum* (or the corresponding substitution of the *definiendum*).

(*g*) Given a definition whose *definiens* (or a substitution of whose *definiens* is negated by a thesis of the type '$\phi\!-\!(\alpha\Lambda)$', we may add a new thesis which negates the *definiendum* (or the corresponding substitution of the *definiendum*).

(*h*) This directive resembles directive η of system $\mathfrak{S}_2$. In each semantic category there can be defined a finite number of basic constants, whose definitions must conform to a certain 'schema characterised in an inductive manner' by the directive[102]. One expression may be described as a basic substitution of another if each variable bound by the main quantifier of the second expression is replaced in the first expression by an appropriate basic constant. Now, given that all basic constants have been defined in the semantic categories of all variables bound by the main quantifier of a generalisation, and given that all possible basic substitutions of this generalisation are already theses of the system, this directive permits us to add the generalisation as a new thesis.

(*i*) If one basic substitution of a generalisation is negated by a thesis of the system, we may add the negation of the generalisation to the system as a new thesis.

Leśniewski briefly contrasts the resulting system with Łukasiewicz's, and makes the following remarks[103]:

(1) The new system has nothing corresponding to Łukasiewicz's signs of assertion and rejection[104].

(2) The primitive terms of Leśniewski's system are the words for implication '$\phi\!-$' and for logical 'null' 'Λ', while Łukasiewicz uses an additional primitive logical 'one'[105].

(3) Łukasiewicz's system is based on three axioms, two of which contain variables. The new system is based on a single axiom containing no variables.

(4) Leśniewski's system, unlike its predecessor, has a directive which explicitly permits definitions to be added to the system.

(5) Unlike Łukasiewicz's system, the new one does not have a substitution directive.

After describing informally the directives of his system of computative protothetic, Leśniewski presents a deduction in the system based on the axiom '$\phi\!-\!(\Lambda\Lambda)$'. The deduction proves on this basis the thesis[106]

$$\llcorner f \lrcorner \ulcorner \phi\!-\!\Big(f\big(f(\llcorner p \lrcorner \ulcorner p \urcorner)\big)\llcorner p \lrcorner \ulcorner f(p) \urcorner\Big)\urcorner$$

[102] ref Leśniewski38, p. 38. According to Słupecki53, p. 80, Leśniewski referred referred to these constants as 'verifiers' in his lectures.

[103] Leśniewski38, p. 36.

[104] Leśniewski gives no reason here for this change, but elsewhere he wrote an extensive attack on assertion signs, charging that their meaning is ambiguous. See Leśniewski27, pp. 170–5.

[105] Leśniewski remarks that we ought also to count among the primitive terms of Łukasiewicz's system his 'defined' terms and his definitional sign of equality. See Leśniewski38, p. 36.

[106] Leśniewski38, p. 41. Cf. the similar thesis in Meredith51, section 1.ii, p. 37.

which is an implicational form of the thesis discovered by Tarski in 1922 and mentioned above in connection with thesis *A3a*. In the next chapter I shall present a similar system as an example of computative protothetic based on equivalence.

3.10. Alternative systems of computative protothetic

Altogether Leśniewski constructed some sixteen systems of computative protothetic based on ten different combinations of primitive terms[107]. Each system has two primitive terms, one a propositional constant, and the other a functor having the Ajdukiewicz index '$\frac{\mathrm{s}}{\mathrm{s\,s}}$'. Almost all of these systems date from 1924. The exceptions date from 1933, and in the table which follows they have the negation forms 'ό(pΛ)', 'ό(Λp)', '-ό-(pV)', and '-ό-(Vp)'[108]. This table gives the primitive terms and the form of negation used in each system, together with an example of a function which could be used to represent definitional equivalence in each system.

$\frac{\mathrm{s}}{\mathrm{s\,s}}$	s	negation	definitions
ό	Λ	ό(pΛ)	ό(ό(pό(qΛ))ό(ό(pΛ)q))
ό	Λ	ό(Λp)	ό(ό(pό(Λq))ό(ό(Λp)q))
ό	Λ	ό(pp)	ό(ό(pό(qq))ό(ό(pp)q))
ό	V	ό(pp)	ό(ό(pό(qq))ό(ό(pp)q))
ϕ	Λ	ϕ(pΛ)	ϕ(pq)
ϕ	Λ	ϕ(Λp)	ϕ(pq)
o-	V	o-(pV)	o-(o-(pq)o-(o-(qp)V))
ϕ-	Λ	ϕ-(pΛ)	ϕ-(ϕ-(ϕ-(pq)ϕ-(ϕ-(qp)Λ))Λ)
-o	V	-o(Vp)	-o(-o(V-o(pq))-o(qp))
-ϕ	Λ	-ϕ(Λp)	-ϕ(Λ-ϕ(-ϕ(Λ-ϕ(pq))-ϕ(qp)))
-o-	V	-o-(pV)	-o-(-o-(pq)V)
-o-	V	-o-(Vp)	-o-(V-o-(pq))
-ό-	Λ	-ό-(pp)	-ό-(-ό-(pq)-ό-(-ό-(pp)-ό-(qq)))
-ό-	V	-ό-(pp)	-ό-(-ό-(pq)-ό-(-ό-(pp)-ό-(qq)))
-ό-	V	-ό-(pV)	-ό-(-ό-(pq)-ό-(-ό-(pV)-ό-(qV)))
-ό-	V	-ό-(Vp)	-ό-(-ό-(pq)-ό-(-ό-(Vp)-ό-(Vq)))

Leśniewski also knew that, like equivalence, inequivalence could be used as a convenient means of expressing definitions[109]. If we use inequivalence to express definitions, the

[107] Leśniewski38, pp. 41–2.

[108] Leśniewski38, p. 42.

[109] I learned of this unpublished remark from Prof. Czesław Lejewski.

form of definitions can be simplified in at least four of the above systems, namely those including ‘ϕ-’, ‘-ϕ’, or ‘-o-’ among their primitive terms.

The directives of each of these systems of computative protothetic are exactly analogous to those of the system explained above. Obviously directives *a–d* must be adjusted to suit the truth conditions of the primitive functor, directives *e–g* must be adjusted to take into account the intended form for definitions, and directives *g*, *i*, and any directives *a–d* which involve negation, must be adjusted to take into account the intended form for negations[110].

In each system, the sole axiom is the shortest true expression which can be constructed out of the primitive terms; that is, in systems having the primitive term ‘V’, the axiom consists of this term, while in systems having the primitive term ‘Λ’, the axiom has the form ‘$\phi(\Lambda\Lambda)$’, where ‘ϕ’ is the primitive functor[111].

Every system of computative protothetic constructed by Leśniewski has exactly two primitive terms. Because he is known to have insisted that the number of primitive terms must be kept to a minimum[112], it is clear that his understanding of computative protothetic did not allow him to substitute for the primitive term ‘Λ’ the expression ‘$\llcorner u \lrcorner \ulcorner u \urcorner$’, even though the truth conditions of both are ordinarily the same.

It is not difficult to see that the ten combinations of primitive terms appearing in the above systems are the only possible bases for systems of computative protothetic, that is, unless we increase either the number of directives[113] or the number of primitive terms[114]. For the primitive terms must be able to express negation and equivalence without using quantifiers. Equivalence cannot be expressed unless the truth value of sentences whose functor is the primitive functor depends in some way on two arguments of the functor; that is, the functors of one argument are insufficient, the truth values of functions whose functors are ‘o’ and ‘-ϕ-’ are not affected by the truth values of the arguments, and those whose functors are ‘ϙ-’, ‘o̍-’, ‘-ϙ’, and ‘-o̍’ are affected by the truth values of just one argument. Finally the functors of conjuntion and alternation, ‘ϙ’ and ‘-ϙ-’, are unable to express either negation or equivalence with the help of any single propositional constant.

In 1934 Leśniewski mentioned in his lectures that systems of computative protothetic containing the term ‘V’ among their primitive terms

> have a rather troublesome property from the point of view of the harmony of these systems, which is that in them the axiom is the only thesis which cannot be repeated in the system[115].

He suggested two ways to remove this feature: either modify the directives to forbid repeating any thesis, or replace the axiom with a new directive allowing us to add to the system

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[110] Cf. Leśniewski38, pp. 41–2.

[111] Leśniewski38, p. 41.

[112] Sobociński56, p. 58. Cf. also Leśniewski29, p. 11.

[113] For example, we could easily construct systems of computative protothetic based on two terms, one of which is a sentence-forming functor which requires three sentences as arguments.

[114] It is possible to construct a system of computative protothetic based on three independent primitive terms. Cf. Lejewski68.

[115] Leśniewski38,pp. 42–3.

as a new thesis any expression equiform with the deleted axiom. The latter method is of interest because the resulting system of protothetic has no axiom at all.

In general, the various alternative systems of computative protothetic provide us with an excellent example of systems whose directives are formulated in such a way that parallel systems can easily be constructed based on different primitive terms[116].

[116] Cf. Leśniewski29, p. 45, and Leśniewski31, p. 292.

4. A System of Computative Protothetic

In this chapter we shall construct a system of computative protothetic based on the primitive terms 'ϕ' and 'Λ'. The directives of this system are stated informally after discussing certain problems that arise in the interpretation of Leśniewski's systems of computative protothetic. The system we construct nevertheless conforms to the formal statement of its directives, which follows later. The formal directives may be understood more easily when the system which they describe is already familiar, and when the restrictions which they incorporate have been grasped by seeing them in action.

4.1. Comments and problems

Any attempt to construct systems of computative protothetic, or similar systems which we may wish to call by some other name, must decide which restrictions should be placed on each of the directives. There is no doubt that variants are possible; for example, I have constructed a system of 'computative protothetic' which does not restrict any semantic category to a finite number of constants; in other words, like $\mathfrak{S}_5$ and unlike $\mathfrak{S}_2$, it allows us to define synonyms which have different shapes. It is our present purpose, however, to construct a system of computative protothetic which resembles as closely as possible those constructed by Leśniewski, and which can be used for similar purposes.

It is most important that the directives of a system of computative protothetic incorporate mechanisms which guarantee that directive *h* for verification is appropriately applied, and that directive *e* for definition allows us to define all the terms we need for directive *h*, and only those terms. The solution which I propose is, I believe, a very plausible one; that is, I think it is probably very close in most details to whatever solution Leśniewski actually formulated privately and discussed informally in Leśniewski38. My solution is based on my understanding of 'automatic verification' and of the purpose of each of the directives.

The style of procedure which Leśniewski calls 'automatic verification' is designed so that in most cases there is only one way to prove or disprove a given expression. Every meaningful expression in protothetic is either a term, a function, or a generalisation[1]. Such an expression is 'decided' when it is proved, or when its negation is proved. With the exception of the axiom and of definitions added in conformity with directive *e*, each kind of meaningful expression is decided in only one way. Generalisations must be proved by directive *h* and disproved by directive *i*. Defined propositional terms must be proved by directive *f* and disproved by directive *g*. Propositional functions whose functor is not the primitive functor must be proved by directive *f* and disproved by directive *g*. Propositional functions whose functor is the primitive functor must be proved or disproved by directives *a*–*d*, depending on how their arguments have been decided; it is clear from the statement of directives *a*–*d* that both of their arguments must have been decided[2]. Given any meaningful expression which has not yet been decided, to disprove it we must first disprove (or

[1] Leśniewski31, p. 301.

[2] Quine suggested simplifying the system of computative protothetic based on implication by combining its directives *a*, *c*, and *d* into one directive, stating that you can add an implication if either its consequent is a thesis, or its antecedent is negated in a thesis; see Quine40, p. 84. A directive of this kind could not easily be adapted to other primitive terms, and would thus destroy the parallel that exists between various systems of computative protothetic. Moreover, the directive

prove) some ‘simpler’ expression, while to prove the given expression we must first prove (or disprove) the same ‘simpler’ expression.

I have said that ‘in most cases’ there is only one way for a given meaningful expression to be decided. I shall state the exceptions explicitly, although some of them are fairly trivial:

(1) The directive for definitions has not been specified explicitly. It is, however, very likely that the ‘schema characterised in an inductive manner’ leaves us free to chose not only which term to define next but also just how we shall define it.

(2) There is often more than one way to prove a given thesis by applying directive i. Thus, for example, assuming that we have proved both that $\phi\!\!-\!\big(-(\Lambda)\Lambda\big)$ and that $\phi\!\!-\!\big(-(\mathrm{V})\Lambda\big)$, and that we wish to prove that $\phi\!\!-\!\big(\llcorner p\lrcorner\ulcorner -(p)\urcorner\Lambda\big)$, then we may apply directive i to either of the former expressions.

(3) After a thesis has been proved, it can often be repeated by appealing to some directive other than the one by which it was first introduced. For example, after proving the thesis $\phi\!\!-\!\big(-(\Lambda)\Lambda\big)$, we can repeat it by using directive d, although we cannot possibly have used directive d to introduce the first thesis of this form.

(4) After a thesis has been repeated, subsequent justifications can appeal to either of the equiform theses without distinguishing between them.

We know that in Leśniewski’s systems the concept of ‘meaningful expression’ grows as the system grows, being extended as new terms are defined. Although an expression such as ‘$\llcorner u\lrcorner\ulcorner u\urcorner$’ has meaning at an early stage of development in systems of computative protothetic, it is clearly unable to replace the primitive term ‘Λ’ in those systems in which that term is primitive. In system $\mathfrak{S}_5$, as in most of Leśniewski’s deductive theories, variables are meaningful not only in any category in which a constant exists, but also in the category of sentences (and of names, if names have meaning in the system). Sentence variables are apparently not allowed in the axioms of computative protothetic, otherwise we could reduce the number of primitive terms. It is likely that a different restriction applies: almost certainly variables are not allowed in any semantic category in a system of computative protothetic until enough basic constants have been introduced to ‘exhaust’ that category. An alternative possibility is that variables are allowed in any semantic category only after at least one constant has appeared in that category. If this policy is adopted, however, a serious problem arises, in that one can define a new term in such a way that some expressions containing it cannot be decided. For example, we might introduce into our equivalential system the definition

$$\llcorner pq\lrcorner\ulcorner\phi\Big(\llcorner fr\lrcorner\ulcorner\phi\Big(\phi\big(f(pr)f(qr)\big)q\Big)\urcorner\varphi(pq)\Big)\urcorner$$

To prove from this definition the law of conjunction which says that $\varphi(\mathrm{VV})$, we must first prove, among others, a thesis which says that $\phi\Big(\phi\big(\varphi(\mathrm{VV})\varphi(\mathrm{VV})\big)\mathrm{V}\Big)$, but we cannot prove this without first proving the required law of conjunction. Hence that law of conjunction can not

formulated by Quine does not require that both arguments of an implication be decided before the implication is decided, and this is contrary to ‘automatic verification’ as I understand it.

be proved on the basis of this definition. It is not easy to allow variables in categories which have not been exhausted and in the same system to prevent definitions from introducing the possibility of meaningful expressions which cannot be decided. For this reason I do not allow variables in any semantic category in the system until constants have been introduced which 'exhaust' the given category. In practice this can be understood as extending the concept of meaningful expression whenever a variable is introduced into a category in which previously no variable has appeared, so that variables in that category become meaningful at that point.

The mechanism for determining that a category has been exhausted is complex, and probably should not be allowed to complicate the definition directive *e*. For this reason I have placed in directive *e* the restriction that variables may appear in a given semantic category only if a previous thesis contains a variable in that category. I believe Leśniewski's systems very likely incorporated a similar restriction. For similar reasons I have restricted directive *i* so that it cannot introduce variables into a semantic category for the first time. One may argue that this is likely to be a departure from Leśniewski's system, but it does somewhat simplify this difficult directive, and thus helps make it somewhat more intelligible. Therefore, in my system only directive *h* can extend the concept of meaningful expressions by introducing variables into a semantic category in which they have not previously occurred.

Because only a finite number of constants can be introduced into any semantic category, computative protothetic does not allow us to define 'synonyms' for constants, and this causes a problem. In $\mathfrak{S}_5$ there is no restriction on the shapes of variables: we can have two words of the same shape in the same thesis, one of which is a constant, and the other of which is a variable[3]. In most proofs we can eliminate any problematic constant by defining a synonym which will replace the constant temporarily during the proof[4]. But this strategy is not available in computative protothetic. For example, to prove an expression containing 'nested' generalisations, such as

$$\llcorner x \lrcorner \ulcorner \phi\Big(\llcorner f \lrcorner \ulcorner f(x) \urcorner\Big) \urcorner$$

we must first prove a series of corresponding theses of the type '$\phi\Big(\llcorner f \lrcorner \ulcorner f(a) \urcorner\Big)$', in each of which the word 'a' represents a different constant in the semantic category of the variable 'x' in the former expression. If the variable 'f' is equiform with any constant 'a' in the semantic category of 'x', then the nested generalisation is not provable, because no corresponding thesis can be proved containing the constant 'a' equiform with the variable 'f'[5]. For this reason computative protothetic must place some restriction on the shapes available for variables. I have chosen the simplest restriction, which is also the most restrictive: no variable in any thesis may have the same shape as any constant. This restriction affects

[3] Cf. remark *A*, Leśniewski29, p. 76.

[4] I noted many years ago that this method cannot be applied to the primitive term in $\mathfrak{S}_5$ when distribution of the quantifier is involved; for example, I do not know of any way to prove in $\mathfrak{S}_5$ that $\phi(\llcorner u \lrcorner \ulcorner u \urcorner \llcorner \phi \lrcorner \ulcorner \phi \urcorner)$. For this reason I suggested in the draught of LeBlanc85, which I submitted for publication in 1979, that the directives of $\mathfrak{S}_5$ should be modified so that no variable might be allowed to have the same shape as the primitive term 'ϕ'. This suggestion, which was peripheral to the main topic of the article, was removed by the editor (without my approval or knowledge) before the article actually appeared some six years later.

[5] Cf. terminological explanation *XIV*, Leśniewski29, p. 65.

directives e, h, and i. It is almost certain that Leśniewski's systems of computative protothetic incorporated the same restriction. The simplest alternative, permitting one variable shape to be substituted for another, is explicitly rejected by Leśniewski[6].

Leśniewski says that the directives h and i of his system of computative protothetic appeal to certain 'basic constants', including the primitive terms;

> and, moreover, a finite number of these shall be defined for each semantic category appearing in the system, according to a schema characterised in advance in an inductive way[7].

To someone unfamiliar with Leśniewski's work, this sounds as though the definition directive states that if definition n has such-and-such a form, then definition $n+1$ has such-and-such a different form. In the light of all of Leśniewski's published directives and explanations, it is most unlikely that the definition directive specifies any definition in its entirety or invokes mathematical induction explicitly. Moreover, the above passage occurs not in the description of the definition directive e, but in the description of the verification directive h.

I believe that the definition directives in Leśniewski's systems of computative protothetic are likely to have been quite similar to the corresponding directives in systems $\mathfrak{S}_4$ and $\mathfrak{S}_5$[8]. To the eighteen conditions specified there, my definition directives add conditions preventing any variable from having the same shape as any term, forbidding variables in any semantic category which has not yet been 'exhausted', and preventing any definition from introducing a synonym for any constant which occurs in the system before the definition is added. These restrictions are sufficient in practice to assure us that the resulting system of computative protothetic is consistent and complete.

The following is the simplest method I can find of determining that a group of constants 'exhaust' a semantic category when appealing to directive h. Every constant in a system of computative protothetic, whether primitive or defined, is characterised by a finite number of *critical expressions*. A critical expression is a term or function in the semantic category of sentences whose first word is the constant being characterised; if it is a function, every argument of every 'parentheme'[9] of the function must be a constant. For example, the critical expressions for the functor of implication are '⫯-(ΛΛ)', '⫯-(ΛV)', '⫯-(VΛ)', and '⫯-(VV)'. A critical expression for one constant is said to *correspond* to a critical expression for another constant whenever every word in the first expression except the first word is equiform with the corresponding word in the second expression. Thus, for example, the expression 'ϙ(ΛV)' corresponds to the second of the above critical expressions for implication.

Corresponding critical expressions are *similar* when they are both proved, or both disproved. Thus, assuming that all of the following critical expressions have been decided appropriately, the expressions '⫯-(ΛΛ)' and 'ϕ(ΛΛ)', like the expressions '⫯-(VV)' and 'ϕ (VV)', are similar, because in each pair both theses are proved when they are decided.

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[6] Remark e, Leśniewski38, p. 36.

[7] Leśniewski38, p. 38.

[8] The directives of system $\mathfrak{S}_4$ were never published. The definition directive for $\mathfrak{S}_5$ appears as terminological explanation *XLIV* in Leśniewski29, pp. 70–2.

[9] 'prntm' in the terminology of Chapter 2. Recall that in Leśniewski's theories a function may have more than one group of parenthesised arguments. Cf. terminological explanation XVIII, Leśniewski29, p. 66.

The expressions '$\phi\!\text{-}(\mathrm{V}\Lambda)$' and '$\phi(\mathrm{V}\Lambda)$' are similar because both are disproved when they are decided. On the other hand, the expressions '$\phi\!\text{-}(\Lambda\mathrm{V})$' and '$\phi(\Lambda\mathrm{V})$' are dissimilar, because the former is proved, while the latter is disproved.

A group of constants exhaust a semantic category when the following conditions are fulfilled[10]:

(1) All possible critical expressions for each constant have been decided.

(2) Given any critical expression A whose first term belongs to the given category, there exists a constant B such that (a) A and a corresponding critical expression for B are dissimilar, and (b) any critical expression which is not equiform with A but begins with the same constant is similar to a corresponding critical expression for B.

I shall present in this chapter a system of computative protothetic based on the primitive terms 'ϕ' and 'Λ', having negations of the type '$\phi(p\Lambda)$' and definitions of the type '$\phi(pq)$'. This combination of primitive terms leads to the simplest possible formulation of directive e for definitions. Moreover, it is easier to compare this system with and to prove it equivalent to a 'standard' system of protothetic having the directives of $\mathfrak{S}_5$, since the two systems have a common primitive term. In this system I shall prove the following theses, which are particularly useful in later discussion:

$$
\begin{array}{ll}
\mathit{C25} & \llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\big(\phi(rq)\phi(pr)\big)\Big) \urcorner \\
\mathit{C26} & \phi(\llcorner u \lrcorner \ulcorner u \urcorner \Lambda) \\
\mathit{C82} & \llcorner f \lrcorner \ulcorner \phi\Big(f\big(f(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\llcorner p \lrcorner \ulcorner f(p) \urcorner\Big) \urcorner \\
\mathit{C94} & \llcorner pq \lrcorner \ulcorner \phi\Big(\phi(pq)\llcorner f \lrcorner \ulcorner \phi\big(f(p)f(q)\big)\urcorner\Big) \urcorner \\
\mathit{C96} & \phi\!\text{-}(\Lambda\Lambda) \\
\mathit{C97} & \phi\!\text{-}(\Lambda\mathrm{V}) \\
\mathit{C98} & \phi\!\text{-}(\mathrm{V}\mathrm{V}) \\
\mathit{C99} & \phi\big(\phi\!\text{-}(\mathrm{V}\Lambda)\Lambda\big)
\end{array}
$$

4.2. Directives

In this section I shall state the directives of my system of computative protothetic informally. The style of the directives resembles that employed to describe Leśniewski's system of implicational computative protothetic in section 3.9, but with more of the necessary conditions stated explicitly. These directives can be understood fully only when they are compared with their formal statement in Chapter 6. Nevertheless, the informal statement of the directives, together with the examples of their application in the next section of this chapter, should make their formal statement easier to follow. There are nine directives:

(a) Given theses of types 'α' and 'β', we can add the corresponding expression of type '$\phi(\alpha\beta)$' to the system as a new thesis.

[10] Conditions 1 and 2, which appear weaker than the more intuitive conditions stated informally at the end of section 4.1, are in fact equivalent to them in the context of the directives of system C, but allow us to verify in a simpler manner that the given constants exhaust the given category.

(*b*) Given theses of types 'α' and '$\phi(\beta\Lambda)$', we can add the corresponding expression of type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$' to the system as a new thesis.

(*c*) Given theses of types '$\phi(\alpha\Lambda)$' and 'β', we can add the corresponding expression of type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$' to the system as a new thesis.

(*d*) Given theses of types '$\phi(\alpha\Lambda)$' and '$\phi(\beta\Lambda)$', we can add the corresponding expression of type '$\phi(\alpha\beta)$' to the system as a new thesis.

(*e*) We may add to the system a definition of the type '$\phi(\alpha\beta)$', or of the same type enclosed in a universal quantifier, provided that it fulfills the eighteen conditions specified in terminological explanation *XLIV* of system $\mathfrak{S}_5$, as well as the following conditions:

(1) No variable in the definition is equiform with any constant in any thesis belonging to the system.

(2) No variable in the definition is equiform with the constant being defined.

(3) The new constant is not equiform with any variable in any thesis belonging to the system.

(4) For every variable in the definition there is a variable in the same semantic category in some thesis already belonging to the system.

(5) The new constant will not be synonymous with any constant in the same category which already occurs in the system. We ensure this by requiring that for each existing constant in that category a critical expression has been decided in such a way that we will be able to prove, with the help of either directive *f* or directive *g*, a dissimilar corresponding critical expression for the newly defined term.

(*f*) Given a definition and another thesis which is equiform with its *definiens* (or which is a correct substitution[11] of its *definiens*), we may add to the system a new thesis equiform with the *definiendum* (or which is the corresponding substitution of the *definiendum*).

(*g*) Given a definition and another thesis which negates an expression equiform with its *definiens* (or which negates an expression which is a correct substitution of its *definiens*), we may add to the system a new thesis which negates an expression equiform with the *definiendum* (or which negates the corresponding substitution of the *definiendum*).

(*h*) Given a generalisation such that each term in its quantifier binds a variable in whose semantic category there exist in the system constants which 'exhaust' that category, and given that the system already contains as a thesis every meaningful substitution of the generalisation in which each variable is replaced by a constant, then we may

[11] As in Leśniewski's system of computative protothetic for implication, there is no rule of substitution. The directives *f* and *g* are restricted forms of what some authors have called 'rules of replacement', since they allow us to infer a thesis containing a defined term from one not containing the term.

add the generalisation to the system as a new thesis. The constants in a semantic category 'exhaust' that category when the following conditions are fulfilled:

(1) For each critical expression A whose first word belongs to the semantic category, there is a constant B in the same category such that

(*i*) Any critical expression for the constant beginning A which is not equiform with A is similar to a corresponding critical expression for B.

(*ii*) A is dissimilar to some corresponding critical expression for B.

(2) There is at least one constant in the semantic category for which all possible critical expressions have been decided.

(*i*) Given a generalisation such that each term in its quantifier binds a variable in a semantic category in which some variable already exists in some thesis already belonging to the system, and given a thesis in the system which disproves an expression which is a substitution of the generalisation in which each variable bound by the main quantifier of the generalisation is replaced by a constant term, we may add to the system a new thesis which negates the given generalisation.

When the system is actually being constructed, we extend it by writing the new theses at the end of the system. In addition to the written theses, which actually belong to the system being constructed, it is customary to write certain other information which does not actually belong to the system. This information includes a unique name for each thesis, which is written to its left, and a justification for adding each thesis, which appears to the right in square brackets.

The justification mentions the directive which enables us to add the thesis in question, then lists the previous theses to which we must refer when we verify that the directive applies in the present case. We shall not list all previous theses even though the directive may require us to examine them, but we shall list theses containing a variable in a given semantic category when these are required by directives e and i, theses which decide critical expressions which are required by directives e and h, and other theses which directive e requires to establish that the term being defined differs from any existing term in the same semantic category. Theses are mentioned only once in a justification, even if the directive appeals to them several times for different purposes.

4.3. The system C

I call this system of computative protothetic system C. It is based on the following axiom:

C1 $\quad \phi(\Lambda\Lambda)$

First we prove some theses which are meaningful relative to axiom *C1*.

C2 $\quad \phi\big(\phi(\Lambda\Lambda)\phi(\Lambda\Lambda)\big)$ $\qquad [a,\ C1]$

C3 $\quad \phi\Big(\phi\big(\phi(\Lambda\Lambda)\Lambda\big)\Lambda\Big)$ $\qquad [b,\ C1]$

C4 $\phi\Big(\phi(\Lambda\Lambda)\phi\big(\phi(\Lambda\Lambda)\phi(\Lambda\Lambda)\big)\Big)$ [*a, C1, C2*]

At this point we extend the concept of meaningful expression in C by defining a new constant, a true sentence.

C5 $\phi\big(\phi(\Lambda\Lambda)\mathrm{V}\big)$ [*e, C1*]

C6 V [*f, C5, C1*]

C7 $\phi(\mathrm{VV})$ [*a, C6*]

C8 $\phi\big(\phi(\mathrm{V}\Lambda)\Lambda\big)$ [*b, C6, C1*]

C9 $\phi\big(\phi(\Lambda\mathrm{V})\Lambda\big)$ [*c, C1, C6*]

C10 $\phi\big(\phi(\mathrm{VV})\phi(\mathrm{VV})\big)$ [*a, C7*]

C11 $\phi\Big(\phi\big(\phi(\mathrm{VV})\Lambda\big)\Lambda\Big)$ [*b, C7, C1*]

C12 $\phi\Big(\phi\big(\phi(\Lambda\Lambda)\phi(\mathrm{V}\Lambda)\big)\Lambda\Big)$ [*b, C1, C8*]

C13 $\phi\Big(\phi\big(\phi(\mathrm{V}\Lambda)\phi(\mathrm{VV})\big)\Lambda\Big)$ [*c, C8, C7*]

C14 $\phi\Big(\phi\big(\phi(\Lambda\mathrm{V})\phi(\Lambda\Lambda)\big)\Lambda\Big)$ [*c, C9, C1*]

C15 $\phi\Big(\phi\big(\phi(\mathrm{VV})\phi(\Lambda\mathrm{V})\big)\Lambda\Big)$ [*b, C7, C9*]

C16 $\phi\big(\phi(\mathrm{V}\Lambda)\phi(\Lambda\mathrm{V})\big)$ [*d, C8, C9*]

C17 $\phi\big(\phi(\Lambda\mathrm{V})\phi(\mathrm{V}\Lambda)\big)$ [*d, C9, C8*]

C18 $\phi\Big(\phi(\mathrm{VV})\phi\big(\phi(\mathrm{VV})\phi(\mathrm{VV})\big)\Big)$ [*a, C7, C10*]

C19 $\phi\Big(\phi(\mathrm{V}\Lambda)\phi\big(\phi(\Lambda\Lambda)\phi(\mathrm{V}\Lambda)\big)\Big)$ [*d, C8, C12*]

C20 $\phi\Big(\phi(\mathrm{V}\Lambda)\phi\big(\phi(\mathrm{V}\Lambda)\phi(\mathrm{VV})\big)\Big)$ [*d, C8, C13*]

C21 $\phi\Big(\phi(\Lambda\mathrm{V})\phi\big(\phi(\Lambda\mathrm{V})\phi(\Lambda\Lambda)\big)\Big)$ [*d, C9, C14*]

C22 $\phi\Big(\phi(\Lambda\mathrm{V})\phi\big(\phi(\mathrm{VV})\phi(\Lambda\mathrm{V})\big)\Big)$ [*d, C9, C15*]

C23 $\phi\Big(\phi(\Lambda\Lambda)\phi\big(\phi(\mathrm{V}\Lambda)\phi(\Lambda\mathrm{V})\big)\Big)$ [*a, C1, C16*]

C24 $\phi\Big(\phi(\mathrm{VV})\phi\big(\phi(\Lambda\mathrm{V})\phi(\mathrm{V}\Lambda)\big)\Big)$ [*a, C7, C17*]

The two propositional constants in fact ‘exhaust’ the semantic category of sentences. Thus we can apply directive h, extending the concept of meaningful expression in system C to allow variables in the semantic category of sentences.

C25 $\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi(pq)\phi\big(\phi(rq)\phi(pr)\big)\Big) \urcorner$ $\big[$*h, C1, C6, C4, C18, C19, C20, C21, C22, C23, C24*$\big]$

Thesis *C25* corresponds to Łukasiewicz's axiom $W12$ for system $\mathfrak{S}$.

C26 $\phi(\llcorner u \lrcorner \ulcorner u \urcorner \Lambda)$ $\big[$*i, C25, C1*$\big]$

In systems of protothetic with the directives of $\mathfrak{S}_5$, a thesis corresponding to *C26* can serve as the definition of 'Λ'.

C27 $\phi(\llcorner u \lrcorner \ulcorner u \urcorner \llcorner u \lrcorner \ulcorner u \urcorner)$ $\big[$*d, C26*$\big]$

C28 $\phi\Big(\phi\big(\phi(\llcorner u \lrcorner \ulcorner u \urcorner \llcorner u \lrcorner \ulcorner u \urcorner)\Lambda\big)\Lambda\Big)$ $\big[$*b, C27, C1*$\big]$

The following definition introduces not only a new constant but also a new semantic category, which contains expressions with the Ajdukiewicz index '$\frac{\mathrm{s}}{\mathrm{s}}$'.

C29 $\llcorner p \lrcorner \ulcorner \phi\big(p \dashv (p)\big) \urcorner$ $\big[$*e, C25*$\big]$

C30 $\phi\big(\dashv(\Lambda)\Lambda\big)$ $\big[$*g, C29, C1*$\big]$

C31 $\dashv(\mathrm{V})$ $\big[$*f, C29, C6*$\big]$

C32 $\phi\big(\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)\Lambda\big)$ $\big[$*g, C29, C26*$\big]$

C33 $\phi\big(\dashv(\Lambda)\dashv(\Lambda)\big)$ $\big[$*d, C30*$\big]$

C34 $\phi\big(\llcorner p \lrcorner \ulcorner \dashv(p) \urcorner \Lambda\big)$ $\big[$*i, C25, C30*$\big]$

C35 $\phi\big(\dashv(\mathrm{V})\dashv(\mathrm{V})\big)$ $\big[$*a, C31*$\big]$

C36 $\phi\Big(\phi\big(\dashv(\mathrm{V})\dashv(\Lambda)\big)\Lambda\Big)$ $\big[$*b, C31, C30*$\big]$

C37 $\phi\Big(\phi\big(\dashv(\Lambda)\dashv(\mathrm{V})\big)\Lambda\Big)$ $\big[$*c, C30, C31*$\big]$

C38 $\phi\Big(\dashv\big(\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\Lambda\Big)$ $\big[$*g, C29, C32*$\big]$

C39 $\phi\bigg(\phi\Big(\phi\big(\dashv(\Lambda)\dashv(\Lambda)\big)\Lambda\Big)\Lambda\bigg)$ $\big[$*b, C33, C1*$\big]$

C40 $\phi\Big(\phi\big(\dashv(\mathrm{V})\dashv(\mathrm{V})\big)\mathrm{V}\Big)$ $\big[$*a, C35, C6*$\big]$

C41 $\phi\bigg(\phi\Big(\phi\big(\dashv(\Lambda)\dashv(\mathrm{V})\big)\mathrm{V}\Big)\Lambda\bigg)$ $\big[$*c, C37, C6*$\big]$

C42 $\phi\Big(\dashv\big(\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\llcorner p \lrcorner \ulcorner \dashv(p) \urcorner\Big)$ $\big[$*d, C38, C34*$\big]$

Three more definitions, *C43*, *C56*, and *C67*, introduce three additional constants in the same semantic category as '$\dashv$'.

C43	$\llcorner p \lrcorner \ulcorner \phi\big(\phi(pp) \vdash\!\!\dashv (p)\big) \urcorner$	$[e,\ C25,\ C30,\ C1]$
C44	$\vdash\!\!\dashv(\Lambda)$	$[f,\ C43,\ C1]$
C45	$\vdash\!\!\dashv(\mathrm{V})$	$[f,\ C43,\ C7]$
C46	$\vdash\!\!\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)$	$[f,\ C43,\ C27]$
C47	$\phi\big(\vdash\!\!\dashv(\Lambda) \vdash\!\!\dashv(\Lambda)\big)$	$[a,\ C44]$
C48	$\phi\big(\vdash\!\!\dashv(\mathrm{V}) \vdash\!\!\dashv(\mathrm{V})\big)$	$[a,\ C45]$
C49	$\phi\big(\vdash\!\!\dashv(\mathrm{V}) \vdash\!\!\dashv(\Lambda)\big)$	$[a,\ C45,\ C44]$
C50	$\llcorner p \lrcorner \ulcorner \vdash\!\!\dashv(p) \urcorner$	$[h,\ C1,\ C6,\ C44,\ C45]$
C51	$\phi\big(\vdash\!\!\dashv(\llcorner u \lrcorner \ulcorner u \urcorner) \vdash\!\!\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)\big)$	$[a,\ C46]$
C52	$\phi\Big(\phi\big(\vdash\!\!\dashv(\mathrm{V}) \vdash\!\!\dashv(\mathrm{V})\big)\mathrm{V}\Big)$	$[a,\ C48,\ C6]$
C53	$\phi\Bigg(\phi\Big(\phi\big(\vdash\!\!\dashv(\mathrm{V}) \vdash\!\!\dashv(\Lambda)\big)\Lambda\Big)\Lambda\Bigg)$	$[b,\ C49,\ C1]$
C54	$\vdash\!\!\dashv\big(\vdash\!\!\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)\big)$	$[f,\ C43,\ C51]$
C55	$\phi\Big(\vdash\!\!\dashv\big(\vdash\!\!\dashv(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\llcorner p \lrcorner \ulcorner \vdash\!\!\dashv(p) \urcorner\Big)$	$[a,\ C54,\ C50]$
C56	$\llcorner p \lrcorner \ulcorner \phi\big(\phi(p\Lambda) \vdash (p)\big) \urcorner$	$[e,\ C25,\ C8,\ C31,\ C45]$
C57	$\vdash(\Lambda)$	$[f,\ C56,\ C1]$
C58	$\phi\big(\vdash(\mathrm{V})\Lambda\big)$	$[g,\ C56,\ C8]$
C59	$\vdash(\llcorner u \lrcorner \ulcorner u \urcorner)$	$[f,\ C56,\ C26]$
C60	$\phi\big(\vdash(\Lambda) \vdash(\Lambda)\big)$	$[a,\ C57]$
C61	$\phi\big(\vdash(\mathrm{V}) \vdash(\mathrm{V})\big)$	$[d,\ C58]$
C62	$\phi\big(\llcorner p \lrcorner \ulcorner \vdash(p) \urcorner \Lambda\big)$	$[i,\ C25,\ C58]$
C63	$\phi\Big(\phi\big(\vdash(\llcorner u \lrcorner \ulcorner u \urcorner)\Lambda\big)\Lambda\Big)$	$[b,\ C59,\ C1]$
C64	$\phi\Big(\phi\big(\vdash(\mathrm{V}) \vdash(\mathrm{V})\big)\mathrm{V}\Big)$	$[a,\ C61,\ C6]$
C65	$\phi\Big(\vdash\big(\vdash(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\Lambda\Big)$	$[g,\ C56,\ C63]$
C66	$\phi\Big(\vdash\big(\vdash(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\llcorner p \lrcorner \ulcorner \vdash(p) \urcorner\Big)$	$[d,\ C65,\ C62]$

C67 $\llcorner p \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pp)\Lambda\big)-(p)\Big) \urcorner$ [*e, C25, C11, C31, C45, C3, C57*]

C68 $\phi\big(-(\Lambda)\Lambda\big)$ [*g, C67, C3*]

C69 $\phi\big(-(\mathrm{V})\Lambda\big)$ [*g, C67, C11*]

C70 $\phi\big(-(\llcorner u \lrcorner \ulcorner u \urcorner)\Lambda\big)$ [*g, C67, C28*]

C71 $\phi\big(-(\Lambda)-(\Lambda)\big)$ [*d, C68*]

C72 $\phi\big(\llcorner p \lrcorner \ulcorner -(p) \urcorner \Lambda\big)$ [*i, C25, C68*]

C73 $\phi\big(-(\mathrm{V})-(\mathrm{V})\big)$ [*d, C69*]

C74 $\phi\big(-(\llcorner u \lrcorner \ulcorner u \urcorner)-(\llcorner u \lrcorner \ulcorner u \urcorner)\big)$ [*d, C70*]

C75 $\phi\Big(\phi\big(-(\mathrm{V})-(\mathrm{V})\big)\mathrm{V}\Big)$ [*a, C73, C6*]

C76 $\phi\Bigg(\phi\Big(\phi\big(-(\llcorner u \lrcorner \ulcorner u \urcorner)-(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\Lambda\Big)\Lambda\Bigg)$ [*b, C74, C1*]

C77 $\phi\Big(-\big(-(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\Lambda\Big)$ [*g, C67, C76*]

C78 $\phi\Big(-\big(-(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\llcorner p \lrcorner \ulcorner -(p) \urcorner\Big)$ [*d, C77, C72*]

At this point we use directive h to extend the concept of meaningful expressions to include variables in the semantic category of ‘$\frac{\mathrm{S}}{\mathrm{S}}$’ functors.

C79 $\llcorner f \lrcorner \ulcorner \phi\big(f(\Lambda)f(\Lambda)\big) \urcorner$
[*h, C30, C31, C44, C45, C57, C58, C68, C69, C33, C47, C60, C71*]

C80 $\llcorner f \lrcorner \ulcorner \phi\big(f(\mathrm{V})f(\mathrm{V})\big) \urcorner$
[*h, C30, C31, C44, C45, C57, C58, C68, C69, C35, C48, C61, C73*]

C81 $\llcorner f \lrcorner \ulcorner \phi\Big(\phi\big(f(\mathrm{V})f(\mathrm{V})\big)\mathrm{V}\Big) \urcorner$
[*h, C30, C31, C44, C45, C57, C58, C68, C69, C40, C52, C64, C75*]

C82 $\llcorner f \lrcorner \ulcorner \phi\Big(f\big(f(\llcorner u \lrcorner \ulcorner u \urcorner)\big)\llcorner p \lrcorner \ulcorner f(p) \urcorner\Big) \urcorner$
[*h, C30, C31, C44, C45, C57, C58, C68, C69, C42, C55, C66, C78*]

Thesis *C82* corresponds to the thesis discovered by Tarski in 1922 which influenced axiom A_h and all subsequent single axioms for system $\mathfrak{S}_5$.

C83 $\phi\Big(\phi(\Lambda\Lambda)\llcorner f \lrcorner \ulcorner \phi\big(f(\Lambda)f(\Lambda)\big) \urcorner\Big)$ [*a, C1, C79*]

C84 $\phi\Big(\llcorner f \lrcorner \ulcorner \phi\big(f(\mathrm{V})f(\Lambda)\big) \urcorner \Lambda\Big)$ [*i, C79, C36*]

C85 $\phi\Big(\llcorner f\lrcorner\ulcorner\phi\big(f(\Lambda)f(\mathrm{V})\big)\urcorner\Lambda\Big)$ [*i*, *C79*, *C37*]

C86 $\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(\Lambda)f(\Lambda)\big)\Lambda\Big)\urcorner\Lambda\Big)$ [*i*, *C79*, *C39*]

C87 $\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(\Lambda)f(\mathrm{V})\big)\mathrm{V}\Big)\urcorner\Lambda\Big)$ [*i*, *C79*, *C41*]

C88 $\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(\mathrm{V})f(\Lambda)\big)\Lambda\Big)\urcorner\Lambda\Big)$ [*i*, *C79*, *C53*]

C89 $\phi\Big(\phi(\mathrm{VV})\llcorner f\lrcorner\ulcorner\phi\big(f(\mathrm{V})f(\mathrm{V})\big)\urcorner\Big)$ [*a*, *C7*, *C80*]

C90 $\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(\mathrm{V})f(\mathrm{V})\big)\mathrm{V}\Big)\urcorner\mathrm{V}\Big)$ [*a*, *C81*, *C6*]

C91 $\phi\Big(\phi(\mathrm{V}\Lambda)\llcorner f\lrcorner\ulcorner\phi\big(f(\mathrm{V})f(\Lambda)\big)\urcorner\Big)$ [*d*, *C8*, *C84*]

C92 $\phi\Big(\phi(\Lambda\mathrm{V})\llcorner f\lrcorner\ulcorner\phi\big(f(\Lambda)f(\mathrm{V})\big)\urcorner\Big)$ [*d*, *C9*, *C85*]

C93 $\phi\Bigg(\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(\mathrm{V})f(\Lambda)\big)\Lambda\Big)\urcorner\mathrm{V}\Big)\Lambda\Bigg)$ [*c*, *C88*, *C6*]

C94 $\llcorner pq\lrcorner\ulcorner\phi\Big(\phi(pq)\llcorner f\lrcorner\ulcorner\phi\big(f(p)f(q)\big)\urcorner\Big)\urcorner$ [*h*, *C1*, *C6*, *C83*, *C89*, *C91*, *C92*]

Thesis *C94* expresses the law of extensionality for sentences.

At this point in the development of system *C* we can prove the four laws of implication after introducing that term by means of a suitable definition.

C95 $\llcorner pq\lrcorner\ulcorner\phi\Bigg(\phi\Big(\llcorner f\lrcorner\ulcorner\phi\Big(\phi\big(f(p)f(q)\big)q\Big)\urcorner p\Big)\phi\text{-}(pq)\Bigg)\urcorner$ [*e*, *C25*, *C9*, *C87*]

C96 $\phi\text{-}(\Lambda\Lambda)$ [*f*, *C95*, *C86*]

C97 $\phi\text{-}(\Lambda\mathrm{V})$ [*f*, *C95*, *C87*]

C98 $\phi\text{-}(\mathrm{VV})$ [*f*, *C95*, *C90*]

C99 $\phi\big(\phi\text{-}(\mathrm{V}\Lambda)\Lambda\big)$ [*g*, *C95*, *C93*]

5. How Leśniewski States Directives

The directives of our system of computative protothetic should conform to Leśniewski's requirements for directives, and if they are to be compared with the directives of $\mathfrak{S}_5$, they should be stated using the same basic terminology.

5.1. General requirements

Leśniewski never gave a systematic account of his requirements for well formalised deductive systems. Sobociński has collected and discussed the requirements for primitive terms and for axioms[1]. We must compile a similar account of the requirements for directives. These, like the requirements for primitive terms and axioms, are not absolute demands but æsthetic requirements towards which we should strive and to which we can appeal when we compare alternative formalisations of a deductive theory.

The directives must be *suited* to the primitive terms of a system. In this sense the usual detachment directive which we adopt in the classical theory of implication or in system $\mathfrak{S}$ is not suited to the Sheffer stroke functor, because if we applied it in such a system, it could lead to inconsistency[2].

Obviously, directives must be *valid*. For Leśniewski validity is more than the simple assurance that a directive will not jeopardise the system's consistency. A truly valid directive must 'bind' him to develop a deductive system only by ways which 'harmonise' with his 'logical intuitions'[3].

The directives should be *adequate* for developing the required system. Adequacy in this sense is related to, but weaker than, the requirement that a deductive system should be complete. In some cases, such as Leśniewski's ontology and mereology, the system is incomplete by design: in these cases a complete system would be less useful than the incomplete system. In other cases, such as system $\mathfrak{S}_1$, we satisfy a special goal by constructing a subsystem of a given theory.

We should be able to *transpose* the directives of one deductive system to another system based on different primitive terms[4]. Leśniewski does not seem to have published any explanation of this requirement, but he says, for example, that in formalising protothetic in 1922

> I took some trouble to formulate the ... directives in such a way that one could easily adapt them to different systems of protothetic independently of the various primitive terms on which these systems would be based[5].

He states elsewhere that anyone who is familiar with the directives of protothetic as based on equivalence should be able 'almost automatically' to understand how to adapt them to

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1 See Sobociński56.
2 Cf. Sobociński56, p. 56.
3 Leśniewski29, p. 78.
4 Leśniewski29, p. 45; cf. Leśniewski38, p. 42.
5 Leśniewski29, p. 14.

produce an equivalent system of protothetic based on another primitive term[6]. Thus he wishes to avoid directives which rely too heavily on special properties of one primitive term. Nicod's detachment directive, for example, relies too heavily on special properties of his primitive term, because it allows us to infer from expressions of the types '$-\!\!\circ\!\!-\big(\alpha -\!\!\circ\!\!-(\beta\gamma)\big)$' and '$\alpha$' the corresponding expression of type 'γ'. The presence of the expression corresponding to 'β' indicates that the directive is in fact stronger than ordinary detachment directives, and this has important consequences in the 'deductive structure' of Nicod's system[7].

The directives of a deductive system should be *independent*[8]. A directive is independent when there is at least one thesis which can be proved only with the help of that directive. Moreover, the theses added by a directive should be independent of each other.

Directives should be as *simple* as possible, because unnecessary complexity makes it more difficult for others to understand the directive[9]. Simpler directives are usually preferable even if they require a more complex axiom system. Łukasiewicz once said that deductive systems should be characterised by axioms rather than by directives[10]. Strictly speaking, this is impossible: even in the classical theory of implication without definitions, the detachment directive plays a highly significant rôle in characterising the primitive term, and in all ordinary deductive systems the substitution directive plays a fundamental rôle in characterising variables.

There should be as *few* directives as possible. If a directive can be replaced by adding a finite number of new axioms to a system, it should be replaced. In general, the resulting system will be easier for others to understand and accept, since it is usually much easier to assure oneself of the truth of an axiom than to verify the validity of a directive.

The directives should be *exclusive* in the sense that they should not refer specifically to any defined term[11]. Thus, for example, in system $\mathfrak{S}_1$, directive 'ζ' was intended to guarantee the equivalence between expressions of the type '$\llcorner x \lrcorner \ulcorner \phi\!\!-\big(\alpha\phi(x)\big)\urcorner$' and corresponding expressions of the type '$\phi\!\!-\big(\alpha \llcorner x \lrcorner \ulcorner \phi(x)\urcorner\big)$'. The constant '$\phi\!\!-$' is not a primitive term of $\mathfrak{S}_1$. It would be simple to formulate the directive by saying that, if any defined term is effectively equivalent to the sign of implication, then certain expressions containing this term would follow legitimately from certain other expressions containing this term. But a directive formulated in this way would violate the requirement of exclusivity.

Leśniewski gave considerable attention to the statement of directives. A directive should be stated in such a fashion that it enables us to decide whether or not a given expression can be added legitimately to a system at a given point in its development. The process of reaching this decision must be finite in length, and it should not require us to examine any expression except the theses of the system and the new expression being examined[12]. In this respect Leśniewski said that

[6] Leśniewski29, p. 45.

[7] Cf. ŁukasiewiczTarski30, p. 36.

[8] Cf. Leśniewski29, p. 39.

[9] Leśniewski29, p. 37.

[10] Łukasiewicz39; McCall67, p. 115; Łukasiewicz70, p. 277.

[11] Leśniewski29, p. 38; cf. Sobociński56, p. 57-8.

[12] Leśniewski31, p. 301.

> among the works which have as their goal the construction of the foundations of mathematics, I really do not know of a single one which establishes, in a way which raises no doubts as to their interpretation, a combination of directives which would suffice for the derivation of all theses actually recognised in the system in question, and which would not at the same time lead to a contradiction in one way or another not foreseen by the author of the given system[13].

The care which Leśniewski gave to formalising his own directives led him to discover defects in the directives, particularly in the definition directives, of deductive systems published by other researchers[14]. As examples of such oversight, he showed how to derive contradictions in the systems of Leon Chwistek and John von Neumann[15].

5.2. Leśniewski's metalanguage

Leśniewski published directives for three deductive systems: $\mathfrak{S}_5$[16], a system of his ontology[17], and a system of the classical 'theory of deduction' with definitions but without quantifiers[18]. The third of these represents the form in which he originally stated directives in his university lectures: there is a series of formal explanations of the terminology which is used to state the directives. These explanations are stated in ordinary language with the help of variables[19]. After each explanation there is an example which satisfies all of the conditions specified in the explanation, and for each condition there is an example which does not satisfy that condition, but which satisfies all of the other conditions in the preceding explanation. Thus the examples show that the various conditions are independent.

When he published Leśniewski29, Leśniewski decided to alter this. There are no examples at all, and the explanations are written using special symbols instead of ordinary words. Directives do not actually belong to the deductive system which they describe, and so should not be expressed in the same symbolic language[20]; therefore he created a different symbolism with which to express the directives and the preliminary explanations of the terms he uses to state the directives. Concerning this he says

> The 'symbolic' formulations given below of the 'terminological explanations' and directives must be regarded simply and solely as typographical abbreviations which would be replaced with corresponding expressions of ordinary language if there were more space at my disposal[21].

The 'symbolic language' of the directives of Leśniewski29 and of Leśniewski30, like the 'ordinary language' of the directives of Leśniewski31, is in fact not only unformalised

[13] Leśniewski29, p. 79.

[14] Leśniewski38, p. 43.

[15] Leśniewski29, pp. 79–81.

[16] Leśniewski29, pp. 63–76.

[17] Leśniewski30, pp. 116–27.

[18] Leśniewski31, pp. 292–309.

[19] Actually, the language of these explanations is quite peculiar: for certain reasons many articles are omitted in defiance of ordinary German idiom, many words have a highly technical usage, footnotes are employed to explain the case of the uninflected variables, and there are some very strange words, such as 'it-is-not-true-that' ('*esistnichtwahrdaß*').

[20] Leśniewski29, p. 59.

[21] Leśniewski29, p. 60.

but also not a formal system. Proofs are constructed in accordance with intuitive criteria, not in accordance with any directives. Any presuppositions which may be required are noted, but they do not form a systematic collection of statements which are in some sense adequate as a basis of a theory. Leśniewski's terminological explanations try, by including certain logically superfluous material, to let us reach certain conclusions without appealing to some rather abstruse linguistic premises.

We know that at some time after 1929, when he used 'symbolic' language in his terminological explanations for the first time, and before his death in 1939, Leśniewski began to use the 'symbolic' explanations in his university lectures[22]. This does not, however, in any sense indicate a fundamental change in his position. Compare, for example, the following explanation, which corresponds to *E7* in the next chapter, but which is stated in the style of his earlier lectures[23]:

I say of an object A that it is the Complex of objects a if, and only if, the following conditions are fulfilled:

(1) A is an expression;

(2) if any object is a word in A, then it is in some a;

(3) if any object B is a, any object C is an a, and some word in B is in C, then B is the same object as C;

(4) if any object is a, then it is an expression in A.

The language used to state terminological explanations and directives is based on Leśniewski's logical intuitions, and thus has much in common with the deductive theories which he constructed to formalise those intuitions. The 'symbolic' form of this 'metalanguage' is based mainly on the symbolism of Peano's *Formulario Matematico* and of Whitehead and Russell's *Principia*.

Many constants correspond to familiar terms in the 'theory of deduction' or in protothetic, for example 'if, and only if', or '$\equiv$', which is used to state definitions. Leśniewski also makes use of properties of these terms when he constructs metalogical proofs, or, more accurately, outlines of proofs, which he presents in a form of 'natural deduction' which he invented, but never explained systematically.

Variables occur only in the semantic category of names. Any variable can be replaced by any proper name, common name, 'empty' name, or name-forming expression. There are no free variables in the 'symbolic' language; every variable must be bound explicitly by a universal or particular quantifier[24]. If Leśniewski expects that a sentence, or part of a sentence, will be true only if a certain variable in it names exactly one object, he uses an upper case letter for the variable; otherwise he uses lower case letters. This convention helps us to understand statements more quickly. All variables are printed in *italic* type.

The term 'is' or 'ε' and several other constants correspond to terms from Leśniewski's ontology. The term 'in' or 'ingredient' from mereology also appears; it is used primarily

[22] I learned of this from Professor Czesław Lejewski.

[23] Cf. especially terminological explanation *I*, Leśniewski31, p. 292.

[24] I have abandoned Leśniewski's practice of using commas between all variables in quantifiers in the Peano/Whitehead/Russell symbolism.

to refer to words or groups of words contained in a sentence. There are also a number of linguistic terms, such as 'word', 'expression', and 'parenthesis', which are also 'primitive' in the sense that they are not defined formally. The 'symbolic' form of these and of all constants not definable in ontology consists of several adjacent letters or digits. All of these constants, that is all but the 'theory of deduction' terms and three ontological 'verbs', are either names or name-forming functors, and the functors take only names as arguments. Different constants may have the same form if they require different numbers of arguments. All constants are printed in Roman type, and as for variables, proper names and functions which are expected to name a single object usually begin with an upper case letter.

The following table lists and provides a way of reading the 'primitive' terms which actually appear in Leśniewski's terminological explanations[25]:

expression	interpretation
$\sim p$	it-is-not-true-that p
$p \equiv q$	p if, and only if, q
$p \supset q$	if p, then q
$p \vee q \vee \ldots$	p or q or $\ldots$
$p.q. \ldots$	p and q and $\ldots$
$[ABC]$	for all A, B, and C
$[\exists ABC]$	for some A, B, and C
$A \varepsilon b$	A is (a) b
$\mathrm{Id}(A)$	the same object as A
$\sim(a)$	object which is not a
$a \cap b \cap \ \ldots$	object which is a and b and $\ldots$
$a \cup b \cup \ \ldots$	object which is a or b or $\ldots$
$a \propto b$	there are as many a as b
$a \backsim b$	there are fewer a than b
$\mathrm{in}(A)$	in A, belonging to A[26]
vrb	word
expr	expression
prnt	parenthesis (left or right)
prnt1	left parenthesis
$\mathrm{prntsym}(A)$	parenthesis symmetrical with A
$\mathrm{cnf}(A)$	expression equiform with A
thp	thesis of this system of protothetic
$\mathrm{prcd}(A)$	object preceding A
$\mathrm{scd}(A)$	object following A
$\mathrm{Uprcd}(A)$	last word preceding A
$\mathrm{Uingr}(A)$	last word in A
$\mathrm{1ingr}(A)$	first word in A
$\mathrm{2ingr}(A)$	second word in A

Any word in an expression can be identified by number in the same fashion; for example, '$\mathrm{53ingr}(A)$' names the fifty-third word in A.

[25] Cf. Leśniewski29, pp. 60–1.

[26] Leśniewski uses the symbol 'ingr' for this term, both in his metalanguage and in mereology; cf. Leśniewski31a, p. 151. I have changed this to 'in' in the metalanguage in order to conform to the current practice in mereology.

In addition to the terms above, Leśniewski uses two others in the examples which demonstrate the independence of each factor of a terminological explanation. Since he never published examples in 'symbolic' form, we must invent new 'symbols' for them. The first is 'complete collection of a', which Leśniewski probably abbreviated as 'Kl(a)', and which we shall write as 'Ccl(a)'[27]. The other is 'one of the mutually equiform expressions "ϕ" ', where 'ϕ' is replaced by some expression from the object language[28]. In this case we shall use the corresponding expression ' "ϕ" ', following the lead of Professor Lejewski in this respect[29].

Terminological explanations, as well as the directives to which they lead, may refer to the axiom of the system in which they appear, but they may not refer to any other theses of the system. Independence examples, on the other hand, may refer to any convenient expression, since these examples are not actually part of the directives, but simply attempt to clarify the explanation which they follow.

When he published his 'symbolic' terminological explanations, Leśniewski always printed the 'logical factors' one under the other to improve perspicuity; our explanations in the next chapter observe this convention. Moreover, we number each factor 1 to n, even if there is only one factor. The independence examples are numbered from 0 to n. In each case example 0 satisfies all specified conditions, and each of the examples 1 to n satisfies all specified conditions except the corresponding factor. The examples given in Leśniewski31 usually contain as well some commentary explaining why the corresponding condition is not fulfilled. In the present work we shall simply assume that the reader understands why the condition is not fulfilled.

5.3. Presuppositions

In 1929 Leśniewski published a series of comments which are intended to prevent anyone who reads his terminological explanations from misunderstanding the terms introduced in the last section[30]. We shall now repeat and expand those remarks.

$[\exists A]$ The particular quantifier is not 'existential' because the variables it contains can be replaced in legitimate substitutions by 'empty' names. Thus from 'Pegasus does not exist' we can infer 'For some A – A does not exist'[31].

$A \varepsilon b$ The expression 'A is b' is true whenever the following conditions are fulfilled:

(1) At least one object is A.

(2) At most one object is A.

(3) Any object which is A is b.

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27 Leśniewski31, p. 293; cf. Leśniewski31a, p. 151, and Lejewski89, p. 481.

28 Leśniewski31, p. 306.

29 Examples of this (using single instead of double quotation marks) appear in Lejewski89; e.g., p. 488.

30 Leśniewski29, pp. 61–2.

31 Cf. Lejewski55a, p. 106.

Id Any object A is the same object as B whenever both A is B and B is A. Note that two equiform objects in different places are not the same object.

∞ There are as many a as b, among other cases, when there are no a and no b. This term can be introduced into Leśniewski's ontology by means of the definition[32]

$$[ab]::[\exists\phi].\therefore[A]:A\ \varepsilon\ a.\supset.[\exists B].B\ \varepsilon\ b.B\ \varepsilon\ \phi(A).\therefore[A]:A\ \varepsilon\ b.\supset.[\exists B].B\ \varepsilon\ a.A\ \varepsilon\ \phi(B).\therefore[ABC]:A\ \varepsilon\ a.B\ \varepsilon\ a.C\ \varepsilon\ b.C\ \varepsilon\ \phi(A).C\ \varepsilon\ \phi(B).\supset.A\ \varepsilon\ B.\therefore[ABC]:A\ \varepsilon\ a.B\ \varepsilon\ b.C\ \varepsilon\ b.B\ \varepsilon\ \phi(A).C\ \varepsilon\ \phi(A).\supset.B\ \varepsilon\ C.\therefore\equiv.a\ \infty\ b$$

but this definition is not acceptable in the metalanguage because it contains a variable functor.

$\rotatebox[origin=c]{-90}{6}$ There are fewer a than b, among other cases, when there are no a and there is at least one b. This term can be introduced into ontology by means of the definition[33]

$$[ab]::[\exists c].\therefore[A]:A\ \varepsilon\ c.\supset.A\ \varepsilon\ b.\therefore a\ \infty\ c::[c].\therefore[A]:A\ \varepsilon\ c.\supset.A\ \varepsilon\ a:\supset.\sim(c\ \infty\ b)::\equiv.a\ \rotatebox[origin=c]{-90}{6}\ b$$

in This term can be taken as the primitive term of mereology. Among its properties are these:

(1) If any object is in A, then A is an object.

(2) If any object B is in C, then any object which is in B is in C.

(3) If any object A is in B, and B is in A, then A is the same object as B.

(4) Any object A is in A.

Ccl This term is used in mereology. Among its properties are

(1) If any object is a, then it is in the complete collection of a.

(2) There is at most one complete collection of a.

(3) If any object A is in the complete collection of b, then some object which is in A is in some object which is b.

Complete collections ordinarily appear in the metalanguage only to refer to objects which consist of words but which are not expressions.

vrb 'Man', 'word', 'p', '$\llcorner$', and ')' are examples of words. The expressions 'the man', '(p)', and '$f_\llcorner$(word' are not words, but are expressions consisting of two, three, and four words respectively. Axiom *C1* consists of five words. Axiom A_o consists of fifty-four words. Letters, dots, marks, or indices which are parts of words are not words.

expr Every word is an expression. The complete collection of a finite number of successive words from any expression is an expression. The complete collection

[32] Cf. Leśniewski29a, p. 98.

[33] Ibid.

consisting of the first, third, and fourth words of some expression is not an expression. If there were a complete collection consisting of an infinite number of words, it would not be an expression.

prnt The words ‘(’, ‘$\Big($’, ‘[’, and ‘{’ are all both parentheses and left parentheses (prnt1). The words ‘)’, ‘]’, and ‘}’ are examples of parentheses which are not left parentheses. The words ‘⌞’ and ‘⌝’ are examples of words which are not parentheses.

prntsym Each of the words ‘(’, ‘$\Big($’, and ‘$\Bigg($’ is a parenthesis symmetrical with each of the words ‘)’, ‘$\Big)$’, and ‘$\Bigg)$’, and conversely. Each of the words ‘}’, ‘$\big\}$’, and ‘$\Big\}$’ is a parenthesis symmetrical with each of the words ‘{’, ‘$\big\{$’, and ‘$\Big\{$’, and conversely. None of the words ‘(’, ‘]’, ‘$\underset{1}{)}$’, and ‘}’ is a parenthesis symmetrical with the word ‘(’.

cnf Every expression is an expression equiform with itself. The first word of *C1* is an expression equiform with the first word of *C2*. The expression called Samp in the next chapter is an expression equiform with thesis *C26*. The word ‘ϕ’ is not equiform with ‘$\underset{1}{\phi}$’; Leśniewski allows two different constants with the same ‘truth conditions’ to share a basic outline. The parenthesis ‘(’ is an expression equiform with the parenthesis ‘$\Big($’, because parentheses are allowed to vary in size for the sake of perspicuity. The parenthesis ‘(’ is not an expression equiform with the parenthesis ‘[’. The word ‘⌜’ is an expression equiform with the word ‘⌜’, because the quantifier scope indicators ‘⌜’ and ‘⌝’ are allowed to vary in height for the sake of perspicuity. Note that equiform expressions in different places are not the same expression. Failure to appreciate this will distort the terminological explanations and the resulting directives.

“. . .” Double quotation marks are an admittedly awkward device for constructing a name for arbitrary expressions. Note that, while names such as ‘*C1* ’ and ‘Samp ’ are proper names, which denote only one object, double quotation marks form common names, which denote all objects equiform with their argument.

thp The system grows in the course of time. At first only the axiom *C1* is a thesis of this system of protothetic, then, one by one, theses *C2*, *C3*, *C4*, and so on, become theses of this system.

prcd Leśniewski uses this term in such a way that the words in any expression are ordered in such a way that transitivity and trichotomy obtain. Moreover the theses in the system are ordered similarly, with axiom *C1* preceding any other thesis, and with every thesis following any theses which belonged to the system already at the time when it was added. One of two expressions precedes another whenever every word in the first expression precedes every word in the second expression. The independence of certain conditions, for example of *E32.1*, requires that words in one thesis should precede any later thesis, but the explanations, at least when applied successfully, require only that theses precede theses, and that some expressions within a given expression precede others.

6. The Directives of Computative Protothetic

In this chapter we shall actually state the directives of compututative protothetic after giving a large number of terminological explanations. Each explanation has the form of an equivalence much like a definition in one of Leśniewski's systems, except that the *definiens* is the right-hand argument of the equivalence, while the *definiendum* is the left-hand argument.

When the *definiens* consists of the conjunction of several conditions, they are printed so that their first characters are aligned[1]. Each condition is numbered, even if there is only one. After each explanation there are a number of examples. The number of each example is the number of the condition which it does not fulfil; it fulfils all other conditions. Each example numbered 0 fulfils all conditions specified in the explanation.

6.1. Standard directives

This section contains the terminological explanations which are essentially identical with those for system $\mathfrak{S}_5$. They define the fundamental terms used to define meaningful expressions: 'tmp', 'qntf', 'gnrl', 'fnct', and 'arg'. Explanations *E1–E4* and many examples refer to the following sample expression, which in fact is equiform with thesis *C26*:

Samp $\phi(\llcorner u \lrcorner \ulcorner u \urcorner \Lambda)$

Explanations *E28–E31* and *E45* refer to the axiom *C1* of the system of computative protothetic whose directives we are preparing to state.

E1 $[A]{:}A\ \varepsilon$ vrb1[2]$.\equiv.{}^1A\ \varepsilon\ \mathrm{cnf}\big(3\mathrm{ingr}(\mathrm{Samp})\big)$

E1.0 $3\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{vrb1}$

E1.1 $5\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{vrb1})$

E2 $[A]{:}A\ \varepsilon\ \mathrm{vrb2}.\equiv.{}^1A\ \varepsilon\ \mathrm{cnf}\big(5\mathrm{ingr}(\mathrm{Samp})\big)$

E2.0 $5\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{vrb2}$

E2.1 $6\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{vrb2})$

E3 $[A]{:}A\ \varepsilon\ \mathrm{vrb3}.\equiv.{}^1A\ \varepsilon\ \mathrm{cnf}\big(6\mathrm{ingr}(\mathrm{Samp})\big)$

E3.0 $6\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{vrb3}$

E3.1 $3\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{vrb3})$

E4 $[A]{:}A\ \varepsilon\ \mathrm{vrb4}.\equiv.{}^1A\ \varepsilon\ \mathrm{cnf}\big(8\mathrm{ingr}(\mathrm{Samp})\big)$

E4.0 $8\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{vrb4}$

[1] This conforms with Leśniewski's practice in Leśniewski29 and in Leśniewski30.

[2] A is a special word of the first kind.

E4.1 $\quad 4\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{vrb4})$

E5 $\quad [A]{:}A\ \varepsilon\ \mathrm{trm}^{3}.{\equiv}.{}^{1}A\ \varepsilon\ \mathrm{vrb}.$

$$\begin{aligned}&{}^{2}A\ \varepsilon \sim\!(\mathrm{prnt}).\\&{}^{3}A\ \varepsilon \sim\!(\mathrm{vrb1}).\\&{}^{4}A\ \varepsilon \sim\!(\mathrm{vrb2}).\\&{}^{5}A\ \varepsilon \sim\!(\mathrm{vrb3}).\\&{}^{6}A\ \varepsilon \sim\!(\mathrm{vrb4})\end{aligned}$$

E5.0 $\quad 4\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{trm}$

E5.1 $\quad \mathrm{Samp}\ \varepsilon \sim\!(\mathrm{trm})$

E5.2 $\quad 2\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{trm})$

E5.3 $\quad 3\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{trm})$

E5.4 $\quad 5\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{trm})$

E5.5 $\quad 6\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{trm})$

E5.6 $\quad 8\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!(\mathrm{trm})$

E6 $\quad [AB]{:}A\ \varepsilon\ \mathrm{int}(B)^{4}.{\equiv}.{}^{1}B\ \varepsilon\ \mathrm{expr}.$

$$\begin{aligned}&{}^{2}A\ \varepsilon\ \mathrm{vrb}.\\&{}^{3}A\ \varepsilon\ \mathrm{in}(B).\\&{}^{4}A\ \varepsilon \sim\!\big(1\mathrm{ingr}(B)\big).\\&{}^{5}A\ \varepsilon \sim\!\big(\mathrm{Uingr}(B)\big)\end{aligned}$$

E6.0 $\quad 2\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{int}(\mathrm{Samp})$

E6.1 $\quad 7\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!\Big(\mathrm{int}\Big(\mathrm{Ccl}\big(\mathrm{trm} \cap \mathrm{in}(\mathrm{Samp})\big)\Big)\Big)$

E6.2 $\quad \mathrm{Samp}\ \varepsilon \sim\!\big(\mathrm{int}(\mathrm{Samp})\big)$

E6.3 $\quad 1\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!\big(\mathrm{int}(\mathrm{C1})\big)$

E6.4 $\quad 1\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\!\big(\mathrm{int}(\mathrm{Samp})\big)$

E6.5 $\quad \mathrm{Uingr}(\mathrm{Samp})\ \varepsilon \sim\!\big(\mathrm{int}(\mathrm{Samp})\big)$

The term defined in the following explanation allows us to analyse an expression into a collection of discrete, successive expressions.

---

[3] A is a term.
[4] A is a word in the interior of B.

E7 $[Aa]::A\ \varepsilon\ \mathrm{Cmpl}(a)$[5]$.\equiv.\therefore{}^{1}\ A\ \varepsilon\ \mathrm{expr}\therefore$

$^{2}[B]:B\ \varepsilon\ \mathrm{vrb}.B\ \varepsilon\ \mathrm{in}(A).\supset.[\exists C].C\ \varepsilon\ a.B\ \varepsilon\ \mathrm{in}(C)\therefore$

$^{3}[BCD]:B\ \varepsilon\ a.C\ \varepsilon\ a.D\ \varepsilon\ \mathrm{vrb}.D\ \varepsilon\ \mathrm{in}(B).D\ \varepsilon\ \mathrm{in}(C).\supset.B\ \varepsilon$ $\mathrm{Id}(C)\therefore$

$^{4}[B]:B\ \varepsilon\ a.\supset.B\ \varepsilon\ \mathrm{expr}\cap\mathrm{in}(A)$

E7.0 $\mathrm{Samp}\ \varepsilon\ \mathrm{Cmpl}\big(\mathrm{vrb}\cap\mathrm{in}(\mathrm{Samp})\big)$

E7.1 $\mathrm{Ccl}\big(\mathrm{trm}\cap\mathrm{in}(\mathrm{Samp})\big)\ \varepsilon\ \sim\!\Big(\mathrm{Cmpl}\big(\mathrm{trm}\cap\mathrm{in}(\mathrm{Samp})\big)\Big)$

E7.2 $\mathrm{Samp}\ \varepsilon\ \sim\!\Big(\mathrm{Cmpl}\big(\mathrm{trm}\cap\mathrm{in}(\mathrm{Samp})\big)\Big)$

E7.3 $\mathrm{Samp}\ \varepsilon\ \sim\!\Big(\mathrm{Cmpl}\big(\mathrm{expr}\cap\mathrm{in}(\mathrm{Samp})\big)\Big)$

E7.4 $\mathrm{1ingr}(\mathrm{Samp})\ \varepsilon\ \sim\!\Big(\mathrm{Cmpl}\big(\mathrm{vrb}\cap\mathrm{in}(\mathrm{Samp})\big)\Big)$

E8 $[A]::A\ \varepsilon\ \mathrm{qntf}$[6]$.\equiv.\therefore{}^{1}\ \mathrm{1ingr}(A)\ \varepsilon\ \mathrm{vrb1}.$

$^{2}\mathrm{Uingr}(A)\ \varepsilon\ \mathrm{vrb2}:$

$^{3}[\exists B].B\ \varepsilon\ \mathrm{int}(A)\therefore$

$^{4}[B]:B\ \varepsilon\ \mathrm{int}(A).\supset.B\ \varepsilon\ \mathrm{trm}\therefore$

$^{5}[BC]:B\ \varepsilon\ \mathrm{int}(A).C\ \varepsilon\ \mathrm{int}(A).B\ \varepsilon\ \mathrm{cnf}(C).\supset.B\ \varepsilon\ \mathrm{Id}(C)$

E8.0 $\mathrm{Cmpl}\big(\mathrm{3ingr}(\mathrm{Samp})\cup\mathrm{4ingr}(\mathrm{Samp})\cup\mathrm{5ingr}(\mathrm{Samp})\big)\ \varepsilon\ \mathrm{qntf}$

E8.1 $[A]:A\ \varepsilon\ \text{“}pq_{\lrcorner}\text{”}.\supset.A\ \varepsilon\ \sim\!(\mathrm{qntf})$

E8.2 $[A]:A\ \varepsilon\ \text{“}{}_{\llcorner}pq\text{”}.\supset.A\ \varepsilon\ \sim\!(\mathrm{qntf})$

E8.3 $[A]:A\ \varepsilon\ \text{“}{}_{\llcorner}\ {}_{\lrcorner}\text{”}.\supset.A\ \varepsilon\ \sim\!(\mathrm{qntf})$

E8.4 $[A]:A\ \varepsilon\ \text{“}{}_{\llcorner}({}_{\lrcorner}\text{”}.\supset.A\ \varepsilon\ \sim\!(\mathrm{qntf})$

E8.5 $[A]:A\ \varepsilon\ \text{“}{}_{\llcorner}pp_{\lrcorner}\text{”}.\supset.A\ \varepsilon\ \sim\!(\mathrm{qntf})$

E9 $[A]\therefore\!:A\ \varepsilon\ \mathrm{sbqntf}$[7]$.\equiv::{}^{1}[\exists B].B\ \varepsilon\ \mathrm{int}(A)::$

$^{2}[B]\therefore B\ \varepsilon\ \mathrm{1ingr}(A).\vee.B\ \varepsilon\ \mathrm{int}(A):\supset.\big(\mathrm{vrb3}\cap\mathrm{in}(A)\cap\mathrm{scd}(B)\big)$ ɕ $\big(\mathrm{vrb4}\cap\mathrm{in}(A)\cap\mathrm{scd}(B)\big)::$

$^{3}[B]\therefore B\ \varepsilon\ \mathrm{int}(A).\vee.B\ \varepsilon\ \mathrm{Uingr}(A):\supset.\big(\mathrm{vrb4}\cap\mathrm{in}(A)\cap\mathrm{prcd}(B)\big)$ ɕ $\big(\mathrm{vrb3}\cap\mathrm{in}(A)\cap\mathrm{prcd}(B)\big)::$

---

[5] A is the complex of a.

[6] A is a quantifier.

[7] A is a subquantifier.

E9.0 $\mathrm{Cmpl}\big(6\mathrm{ingr}(\mathrm{Samp}) \cup 7\mathrm{ingr}(\mathrm{Samp}) \cup 8\mathrm{ingr}(\mathrm{Samp})\big)\ \varepsilon\ \mathrm{sbqntf}$

E9.1 $[A]{:}A\ \varepsilon\ \text{“}\ulcorner\,\urcorner\text{”}.\supset.A\ \varepsilon\ \sim(\mathrm{sbqntf})$

E9.2 $[A]{:}A\ \varepsilon\ \text{“}\ulcorner pq\text{”}.\supset.A\ \varepsilon\ \sim(\mathrm{sbqntf})$

E9.3 $[A]{:}A\ \varepsilon\ \text{“}pq\urcorner\text{”}.\supset.A\ \varepsilon\ \sim(\mathrm{sbqntf})$

E10 $[A]{::}A\ \varepsilon\ \mathrm{gnrl}$[8]$.\equiv.\therefore{}^{1}[\exists B].B\ \varepsilon\ \mathrm{qntf}.B\ \varepsilon\ \mathrm{in}(A).1\mathrm{ingr}(A)\ \varepsilon\ \mathrm{in}(B){:}$

$${}^{2}[\exists B].B\ \varepsilon\ \mathrm{sbqntf}.B\ \varepsilon\ \mathrm{in}(A).\mathrm{Uingr}(A)\ \varepsilon\ \mathrm{in}(B).\therefore$$

$${}^{3}[BC]{:}B\ \varepsilon\ \mathrm{qntf}.B\ \varepsilon\ \mathrm{in}(A).1\mathrm{ingr}(A)\ \varepsilon\ \mathrm{in}(B).C\ \varepsilon\ \mathrm{sbqntf}.C\ \varepsilon\ \mathrm{in}(A)$$
$$.\mathrm{Uingr}(A)\ \varepsilon\ \mathrm{in}(C).\supset.A\ \varepsilon\ \mathrm{Cmpl}(B \cup C)$$

E10.0 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \mathrm{gnrl}$

E10.1 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(3\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \sim(\mathrm{gnrl})$

E10.2 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(8\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \sim(\mathrm{gnrl})$

E10.3 $[A]{:}A\ \varepsilon\ \text{“}\llcorner u \lrcorner u \ulcorner u \urcorner\text{”}.\supset.A\ \varepsilon\ \sim(\mathrm{gnrl})$

Note that the above explanation of a generalisation, when compared with the informal account in section 2.2, is both more restrictive (since it forbids repeating terms in a quantifier) and less restrictive (since it does not yet restrict quantified expressions to terms and functions).

E11 $[AB]{:}A\ \varepsilon\ \mathrm{Qntf}(B)$[9]$.\equiv.{}^{1}B\ \varepsilon\ \mathrm{gnrl}.$

$${}^{2}A\ \varepsilon\ \mathrm{qntf} \cap \mathrm{in}(B).$$

$${}^{3}1\mathrm{ingr}(B)\ \varepsilon\ \mathrm{in}(A)$$

E11.0 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(6\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \mathrm{Qntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\Big)$

E11.1 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(6\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \sim\Big(\mathrm{Qntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big)\Big)\Big)\Big)$

E11.2 $3\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \sim\Big(\mathrm{Qntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\Big)\Big)$

---

8 A is a generalisation.

9 A is the quantifier of B.

E11.3 $[A]{:}A\ \varepsilon\ \text{“}\llcorner f\lrcorner\ulcorner f(\llcorner u\lrcorner\ulcorner u\urcorner)\urcorner\text{”}.\supset.\mathrm{Cmpl}\Big(\mathrm{vrb}\ \cap\ \mathrm{in}(A)\ \cap\ \mathrm{scd}\big(6\mathrm{ingr}(A)\big)\ \cap\ \mathrm{prcd}\big(10\mathrm{ingr}(A)\big)\Big)\ \varepsilon\ {\sim}\big(\mathrm{Qntf}(A)\big)$

E12 $[AB]{:}A\ \varepsilon\ \mathrm{Sbqntf}(B)$[10]$.\equiv.{}^1B\ \varepsilon\ \mathrm{gnrl}.$

$^2A\ \varepsilon\ \mathrm{sbqntf}.$

$^3A\ \varepsilon\ \mathrm{in}(B).$

$^4\mathrm{Uingr}(B)\ \varepsilon\ \mathrm{in}(A)$

E12.0 $\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in(Samp)}\cap\mathrm{scd}\big(5\mathrm{ingr(Samp)}\big)\cap\mathrm{prcd}\big(9\mathrm{ingr(Samp)}\big)\Big)\ \varepsilon\ \mathrm{Sbqntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in(Samp)}\cap\mathrm{scd}\big(2\mathrm{ingr(Samp)}\big)\cap\mathrm{prcd}\big(9\mathrm{ingr(Samp)}\big)\Big)\Big)$

E12.1 $\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in(Samp)}\cap\mathrm{scd}\big(5\mathrm{ingr(Samp)}\big)\cap\mathrm{prcd}\big(9\mathrm{ingr(Samp)}\big)\Big)\ \varepsilon\ {\sim}\Big(\mathrm{Sbqntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in(Samp)}\cap\mathrm{prcd}\big(9\mathrm{ingr(Samp)}\big)\Big)\Big)\Big)$

E12.2 $8\mathrm{ingr(Samp)}\ \varepsilon\ {\sim}\Big(\mathrm{Sbqntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in(Samp)}\cap\mathrm{scd}\big(2\mathrm{ingr(Samp)}\big)\cap\mathrm{prcd}\big(9\mathrm{ingr(Samp)}\big)\Big)\Big)\Big)$

E12.3 $[A]{:}A\ \varepsilon\ \text{“}\llcorner f\lrcorner\ulcorner f(\llcorner u\lrcorner\ulcorner u\urcorner)\urcorner\text{”}.\supset.\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in}(A)\cap\mathrm{scd}\big(3\mathrm{ingr}(A)\big)\Big)\ \varepsilon\ {\sim}\Big(\mathrm{Sbqntf}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb}\cap\mathrm{in}(A)\cap\mathrm{scd}\big(6\mathrm{ingr}(A)\big)\cap\mathrm{prcd}\big(13\mathrm{ingr}(A)\big)\Big)\Big)\Big)$

E12.4 $[A]{:}A\ \varepsilon\ \text{“}\llcorner f\lrcorner\ulcorner f(\llcorner u\lrcorner\ulcorner u\urcorner)\urcorner\text{”}.\supset.\mathrm{Cmpl}\Big(\mathrm{vrb}\ \cap\ \mathrm{in}(A)\ \cap\ \mathrm{scd}\big(9\mathrm{ingr}(A)\big)\ \cap\ \mathrm{prcd}\big(13\mathrm{ingr}(A)\big)\Big)\ \varepsilon\ {\sim}\big(\mathrm{Sbqntf}(A)\big)$

The following explanation of the ‘essential part’ or ‘nucleus’ of an expression allows us to refer conveniently to expressions which may or may not be generalisations.

E13 $[AB].\therefore A\ \varepsilon\ \mathrm{Essnt}(B).\equiv{:}{}^1A\ \varepsilon\ \mathrm{Cmpl}\Big(\mathrm{int}\big(\mathrm{Sbqntf}(B)\big)\Big).\vee.A\ \varepsilon\ \mathrm{expr}.A\ \varepsilon\ \mathrm{Id}(B).A\ \varepsilon\ {\sim}(\mathrm{gnrl})$

E13.0 $7\mathrm{ingr(Samp)}\ \varepsilon\ \mathrm{Essnt}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb}\ \cap\ \mathrm{in(Samp)}\ \cap\ \mathrm{scd}\big(2\mathrm{ingr(Samp)}\big)\ \cap\ \mathrm{prcd}\big(9\mathrm{ingr(Samp)}\big)\Big)\Big)$

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10 A is the subquantifier of B.

E13.1 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big) \,\varepsilon \sim\Bigg(\mathrm{Essnt}$
$\bigg(\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\bigg)\Bigg)$

E14 $[ABC]{::}A \,\varepsilon\, \mathrm{var}(B,C)$[11] $.{\equiv}.{\therefore}{}^{1}B \,\varepsilon\, \mathrm{int}\big(\mathrm{Qntf}(C)\big).$
$^{2}A \,\varepsilon\, \mathrm{cnf}(B).$
$^{3}A \,\varepsilon\, \mathrm{in}\big(\mathrm{Essnt}(C)\big){\therefore}$
$^{4}[DE]{:}D \,\varepsilon\, \mathrm{in}(C).E \,\varepsilon\, \mathrm{int}\big(\mathrm{Qntf}(D)\big).A \,\varepsilon\, \mathrm{cnf}(E).{\supset}.D \,\varepsilon$
$\mathrm{Id}(C)$

E14.0 $7\mathrm{ingr}(\mathrm{Samp}) \,\varepsilon\, \mathrm{var}\Bigg(4\mathrm{ingr}(\mathrm{Samp}),\mathrm{Cmpl}\bigg(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big)$
$\cap\ \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\bigg)\Bigg)$

E14.1 $1\mathrm{ingr}(\mathrm{Samp}) \,\varepsilon \sim\Big(\mathrm{var}\big(1\mathrm{ingr}(\mathrm{Samp}), 1\mathrm{ingr}(\mathrm{Samp})\big)\Big)$

E14.2 $[A]{:}A \,\varepsilon\, \text{“}{\llcorner} p {\lrcorner}{\ulcorner}\phi(pp){\urcorner}\text{”}.{\supset}.5\mathrm{ingr}(A) \,\varepsilon \sim\Big(\mathrm{var}\big(2\mathrm{ingr}(A), A\big)\Big)$

E14.3 $4\mathrm{ingr}(\mathrm{Samp}) \,\varepsilon \sim\Bigg(\mathrm{var}\bigg(4\mathrm{ingr}(\mathrm{Samp}),\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp}$
$)\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\bigg)\Bigg)$

E14.4 $[A]{:}A \,\varepsilon\, \text{“}{\llcorner} f {\lrcorner}{\ulcorner}f({\llcorner} f {\lrcorner}{\ulcorner}f{\urcorner}){\urcorner}\text{”}.{\supset}.11\mathrm{ingr}(A) \,\varepsilon \sim\Big(\mathrm{var}\big(2\mathrm{ingr}(A), A\big)\Big)$

The above explanation refers to a variable bound by term B in generalisation C. The following explanation refers to two ‘co-variables’, which are related to each other by being bound by the same term in the same quantifier.

E15 $[ABC]{\therefore}A \,\varepsilon\, \mathrm{cnvar}(B,C)$[12] $.{\equiv}{:}{}^{1}[\exists D].A \,\varepsilon\, \mathrm{var}(D,C){:}$
$^{2}[\exists D].B \,\varepsilon\, \mathrm{var}(D,C){:}$
$^{3}A \,\varepsilon\, \mathrm{cnf}(B)$

E15.0 $7\mathrm{ingr}(\mathrm{Samp}) \,\varepsilon\, \mathrm{cnvar}\bigg(7\mathrm{ingr}(\mathrm{Samp}),\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp}$
$)\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\bigg)$

---

[11] A is a variable bound by B in C.

[12] A is a variable related to B in C.

E15.1 $4\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\Big(\mathrm{cnvar}\Big(7\mathrm{ingr}(\mathrm{Samp}),\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\Big)\Big)$

E15.2 $7\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\Big(\mathrm{cnvar}\Big(4\mathrm{ingr}(\mathrm{Samp}),\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(2\mathrm{ingr}(\mathrm{Samp})\big) \cap \mathrm{prcd}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)\Big)\Big)$

E15.3 $[A]{:}A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \phi(pq) \urcorner$”$.\supset.8\mathrm{ingr}(A)\ \varepsilon \sim\Big(\mathrm{cnvar}\big(9\mathrm{ingr}(A), A\big)\Big)$

E16 $[A]{\therefore}A\ \varepsilon\ \mathrm{prntm}^{13}.\equiv{::}^{1}[\exists B].B\ \varepsilon\ \mathrm{int}(A){::}$

$^{2}[B]{.\!\!:}.B\ \varepsilon\ 1\mathrm{ingr}(A).\vee.B\ \varepsilon\ \mathrm{int}(A){:}\supset.\Big(\mathrm{in}(A) \cap \mathrm{scd}(B) \cap \mathrm{cnf}\big(1\mathrm{ingr}(A)\big)\Big) \circlearrowleft \Big(\mathrm{in}(A) \cap \mathrm{scd}(B) \cap \mathrm{prntsym}\big(1\mathrm{ingr}(A)\big)\Big){::}$

$^{3}[B]{.\!\!:}.B\ \varepsilon\ \mathrm{int}(A).\vee.B\ \varepsilon\ \mathrm{Uingr}(A){:}\supset.\Big(\mathrm{in}(A) \cap \mathrm{prcd}(B) \cap \mathrm{prntsym}\big(1\mathrm{ingr}(A)\big)\Big) \circlearrowleft \Big(\mathrm{in}(A) \cap \mathrm{prcd}(B) \cap \mathrm{prnt1} \cap \mathrm{cnf}\big(1\mathrm{ingr}(A)\big)\Big)$

E16.0 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \mathrm{prntm}$

E16.1 $[A]{:}A\ \varepsilon$ “( )”$.\supset.A\ \varepsilon \sim(\mathrm{prntm})$

E16.2 $\mathrm{Cmpl}\big(\mathrm{int}(\mathrm{Samp})\big)\ \varepsilon \sim(\mathrm{prntm})$

E16.2 $[A]{:}A\ \varepsilon$ “( )$\big)$”$.\supset.A\ \varepsilon \sim(\mathrm{prntm})$

Note that the explanation of a bracketed expression, when compared with the informal account in section 2.2, is both more restrictive (since the initial and terminal parentheses must be of the same kind) and less restrictive (since it does not yet specify what kinds of expression are permitted between the parentheses).

E17 $[ABa]{::}A\ \varepsilon\ \mathrm{prntm}(B,a)^{14}.\equiv{.\!\!:}.^{1}\ [C]{:}C\ \varepsilon\ a.\supset.C\ \varepsilon\ \mathrm{prntm}{.\!\!:}.$

$^{2}B\ \varepsilon\ \mathrm{Cmpl}\big(1\mathrm{ingr}(B) \cup a\big).$

$^{3}1\mathrm{ingr}(B)\ \varepsilon\ \mathrm{trm}.$

$^{4}A\ \varepsilon\ a$

E17.0 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \mathrm{prntm}\big(\mathrm{Samp},\mathrm{prntm} \cap \mathrm{in}(\mathrm{Samp})\big)$

---

13 A is a bracketed expression.

14 A is a bracketed expression in B by means of a.

E17.1 $\text{1ingr(Samp)}\ \varepsilon \sim\Big(\text{prntm}\Big(\text{Samp,vrb} \cap \text{in(Samp)}\Big)\Big)$

E17.2 $\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{1ingr(Samp)}\Big)\Big)\ \varepsilon \sim\Big(\text{prntm(Samp,prntm)}\Big)$

E17.3 $[A]{:}A\ \varepsilon$ "$)(p)$"$.\supset.\text{Cmpl}\Big(\text{vrb} \cap \text{in}(A) \cap \text{scd}\Big(\text{1ingr}(A)\Big)\Big)\ \varepsilon \sim\Big(\text{prntm}\Big(A,\text{prntm} \cap \text{in}(A)\Big)\Big)$

E17.4 $\text{Samp}\ \varepsilon \sim\Big(\text{prntm}\Big(\text{Samp,prntm} \cap \text{in(Samp)}\Big)\Big)$

E18 $[AB]{:}A\ \varepsilon\ \text{prntm}(B).\equiv.{}^1[\exists a].A\ \varepsilon\ \text{prntm}(B,a)$

E18.0 $\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{1ingr(Samp)}\Big)\Big)\ \varepsilon\ \text{prntm(Samp)}$

E18.1 $\text{Samp}\ \varepsilon \sim\Big(\text{prntm(Samp)}\Big)$

E19 $[A]{:}A\ \varepsilon\ \text{fnct}$[15]$.\equiv.{}^1[\exists B].B\ \varepsilon\ \text{prntm}(A)$

E19.0 $\text{Samp}\ \varepsilon\ \text{fnct}$

E19.1 $\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{1ingr(Samp)}\Big)\Big)\ \varepsilon \sim(\text{fnct})$

The above explanation of a function, when compared with the informal account in section 2.2, is both more restrictive (since it places further conditions on parentheses) and less restrictive (since it does not yet require that a function have sensible arguments).

E20 $[ABa]{::}A\ \varepsilon\ \text{arg}(B,a)$[16]$.\equiv{::}{}^1B\ \varepsilon\ \text{prntm}{::}$

$^2[C]{\therefore}C\ \varepsilon\ a.\supset{:}C\ \varepsilon\ \text{trm}.\lor.C\ \varepsilon\ \text{gnrl}.\lor.C\ \varepsilon\ \text{fnct}{::}$

$^3\text{Cmpl}\Big(\text{int}(B)\Big)\ \varepsilon\ \text{Cmpl}(a).$

$^4A\ \varepsilon\ a$

E20.0 $\text{9ingr(Samp)}\ \varepsilon\ \text{arg}\Bigg(\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{1ingr(Samp)}\Big)\Big),\text{9ingr(Samp)} \cup \Big(\text{gnrl} \cap \text{in(Samp)}\Big)\Bigg)$

E20.1 $\text{4ingr(Samp)}\ \varepsilon \sim\Bigg(\text{arg}\Bigg(\text{Cmpl}\Bigg(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{2ingr(Samp)}\Big) \cap \text{prcd}\Big(\text{6ingr(Samp)}\Big)\Bigg),\text{4ingr(Samp)}\Bigg)\Bigg)$

E20.2 $\text{9ingr(Samp)}\ \varepsilon \sim\Bigg(\text{arg}\Bigg(\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{1ingr(Samp)}\Big)\Big),\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\Big(\text{2ingr(Samp)}\Big) \cap \text{prcd}\Big(\text{Uingr(Samp)}\Big)\Bigg)\Bigg)$

---

[15] A is a function.

[16] A is an argument of B by means of a.

E20.3 $\quad$ 9ingr(Samp) $\varepsilon \sim\Big(\text{arg}\Big(\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\big(\text{1ingr(Samp)}\big)\Big),\text{trm}\Big)\Big)$

E20.4 $\quad$ Samp $\varepsilon \sim\Big(\text{arg}\Big(\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\big(\text{1ingr(Samp)}\big)\Big), \text{9ingr(Samp)} \cup \big(\text{gnrl} \cap \text{in(Samp)}\big)\Big)\Big)$

E21 $\quad [AB]{:}A\ \varepsilon\ \text{arg}(B).{\equiv}.^1[\exists a].A\ \varepsilon\ \text{arg}(B,a)$

E21.0 $\quad$ 9ingr(Samp) $\varepsilon\ \text{arg}\Big(\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \sim\big(\text{1ingr(Samp)}\big)\Big)\Big)$

E21.1 $\quad$ 7ingr(Samp) $\varepsilon \sim\Big(\text{arg}\Big(\text{Cmpl}\Big(\text{vrb} \cap \text{in(Samp)} \cap \text{scd}\big(\text{1ingr(Samp)}\big)\Big)\Big)\Big)$

The following explanation effectively defines the functor of a function as that part of the function which precedes the final bracketed expression.

E22 $\quad [AB]{:}A\ \varepsilon\ \text{Sgnfnct}(B)^{17}.{\equiv}.^1A\ \varepsilon\ \text{expr}.$

$^2A\ \varepsilon\ \text{in}(B).$

$^3\text{Cmpl}\Big(\text{vrb} \cap \text{in}(B) \cap \sim\big(\text{in}(A)\big)\Big)\ \varepsilon\ \text{prntm}(B)$

E22.0 $\quad$ 1ingr(Samp) ε Sgnfnct(Samp)

E22.1 $\quad [A]{:}A\ \varepsilon\ \text{“}\underset{1\ 1}{\phi(p)(q)}\text{”}^{18}.{\supset}.\text{Ccl}\Big(\text{1ingr}(A) \cup \Big(\text{vrb} \cap \text{in}(A) \cap \text{scd}\big(\text{4ingr}(A)\big)\Big)\Big)\ \varepsilon \sim\big(\text{Sgnfnct}(A)\big)$

E22.2 $\quad [A]{:}A\ \varepsilon\ \text{“}p\phi(pq)\text{”}.{\supset}.\text{Cmpl}\big(\text{1ingr}(A) \cup \text{2ingr}(A)\big)\ \varepsilon \sim\Big(\text{Sgnfnct}\Big(\text{Cmpl}\Big(\text{vrb} \cap \text{in}(A) \cap \text{scd}\big(\text{1ingr}(A)\big)\Big)\Big)\Big)$

E22.3 $\quad$ Uingr(Samp) $\varepsilon \sim\big(\text{Sgnfnct(Samp)}\big)$

E23 $\quad [AB]{:}A\ \varepsilon\ \text{simprntm}(B)^{19}.{\equiv}.^1A\ \varepsilon\ \text{prntm}.$

$^2B\ \varepsilon\ \text{prntm}.$

$^3\text{1ingr}(A)\ \varepsilon\ \text{cnf}\big(\text{1ingr}(B)\big).$

$^4\text{arg}(A)\ \infty\ \text{arg}(B)$

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[17] A is the functor of B.

[18] This is an example of a ‘many-link’ or ‘multi-link’ function, in which one function serves as the functor of another function.

[19] A is a bracketed expression similar to B.

E23.0 $\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{Samp})\big)\Big)\ \varepsilon\ \mathrm{simprntm}\Big(\mathrm{Cmpl}\big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{C1}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{C1})\big)\big)\Big)$

E23.1 $[AB]{:}A\ \varepsilon$ “$(pq$”$.B\ \varepsilon$ “$(\llcorner)$”$.\supset.A\ \varepsilon \sim\big(\mathrm{simprntm}(B)\big)$

E23.2 $[AB]{:}A\ \varepsilon$ “$(\llcorner)$”$.B\ \varepsilon$ “$(pq$”$.\supset.A\ \varepsilon \sim\big(\mathrm{simprntm}(B)\big)$

E23.3 $[AB]{:}A\ \varepsilon$ “(p)”$.B\ \varepsilon$ “$\{p\}$”$.\supset.A\ \varepsilon \sim\big(\mathrm{simprntm}(B)\big)$

E23.4 $[AB]{:}A\ \varepsilon$ “(p)”$.B\ \varepsilon$ “(pp)”$.\supset.A\ \varepsilon \sim\big(\mathrm{simprntm}(B)\big)$

E24 $[AB]{::}A\ \varepsilon\ \mathrm{genfnct}(B)$[20]$.\equiv\therefore{}^1A\ \varepsilon\ \mathrm{fnct}{:}$

$^2\mathrm{prntm}(A)\ \infty\ \mathrm{prntm}(B).\vee.\mathrm{prntm}(A)\ \backsim\ \mathrm{prntm}(B).\therefore$

$^3[CD]{:}C\ \varepsilon\ \mathrm{prntm}(A).D\ \varepsilon\ \mathrm{prntm}(B).\big(\mathrm{prntm}(A) \cap \mathrm{scd}(C)\big)\ \infty\ \big(\mathrm{prntm}(B) \cap \mathrm{scd}(D)\big).\supset.C\ \varepsilon\ \mathrm{simprntm}(D)$

E24.0 $[A]{:}A\ \varepsilon$ “$\phi\underset{2}{(}p\underset{2}{)}(qr)$”$.\supset.\mathrm{Samp}\ \varepsilon\ \mathrm{genfnct}(A)$

E24.1 $9\mathrm{ingr}(\mathrm{Samp})\ \varepsilon \sim\Big(\mathrm{genfnct}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)$

E24.2 $\mathrm{Samp}\ \varepsilon \sim\Big(\mathrm{genfnct}\big(9\mathrm{ingr}(\mathrm{Samp})\big)\Big)$

E24.3 $[A]{:}A\ \varepsilon$ “$\dashv(p)$”$.\supset.A\ \varepsilon \sim\big(\mathrm{genfnct}(\mathrm{Samp})\big)$

E25 $[ABCD]{:}A\ \varepsilon\ \mathrm{Anarg}(B, C, D)$[21]$.\equiv.{}^1C\ \varepsilon\ \mathrm{simprntm}(D).$

$^2A\ \varepsilon\ \mathrm{arg}(C).$

$^3B\ \varepsilon\ \mathrm{arg}(D).$

$^4\big(\mathrm{arg}(C) \cap \mathrm{prcd}(A)\big)\ \infty\ \big(\mathrm{arg}(D) \cap \mathrm{prcd}(B)\big)$

E25.0 $4\mathrm{ingr}(\mathrm{C1})\ \varepsilon\ \mathrm{Anarg}\Big(9\mathrm{ingr}(\mathrm{Samp}),\mathrm{Cmpl}\big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{C1}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{C1})\big)\big),\mathrm{Cmpl}\big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{Samp}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{Samp})\big)\big)\Big)$

E25.1 $[A]{:}A\ \varepsilon$ “(p)”$.\supset.2\mathrm{ingr}(A)\ \varepsilon \sim\Big(\mathrm{Anarg}\Big(3\mathrm{ingr}(\mathrm{C1}), A,\mathrm{Cmpl}\big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{C1}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{C1})\big)\big)\Big)\Big)$

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[20] A is a function modelled after B.

[21] A is the argument corresponding to B in C and D respectively.

E25.2 $[A]{:}A\ \varepsilon$ "(pq)"$.\supset.$3ingr(Samp) $\varepsilon \sim\Big($Anarg$\Big($2ingr(A),Cmpl(vrb $\cap$ in(Samp) $\cap$ scd$\big($1ingr(Samp)$\big)\big)$, $A\Big)\Big)$

E25.3 $[A]{:}A\ \varepsilon$ "(pq)"$.\supset.$2ingr$(A)\ \varepsilon \sim\Big($Anarg$\Big($3ingr(Samp), A,Cmpl$\Big($vrb $\cap$ in(Samp) $\cap$ scd$\big($1ingr(Samp)$\big)\Big)\Big)\Big)$

E25.4 3ingr(C1) $\varepsilon \sim\Big($Anarg$\Big($9ingr(Samp),Cmpl$\Big($vrb $\cap$ in(C1) $\cap$ scd$\big($1ingr(C1)$\big)\Big)$, Cmpl$\Big($vrb $\cap$ in(Samp) $\cap$ scd$\big($1ingr(Samp)$\big)\Big)\Big)\Big)$

E26 $[ABCD]\therefore A\ \varepsilon$ Ansgnfnct(B,C,D)[22]$.\equiv{:}^1A\ \varepsilon$ Sgnfnct(C).
$^2B\ \varepsilon$ Sgnfnct(D):
$^3[\exists EF].E\ \varepsilon$ prntm$(C).E\ \varepsilon$ scd$(A).F\ \varepsilon$ prntm $(D).F\ \varepsilon$ scd$(B).E\ \varepsilon$ simprntm(F)

E26.0 1ingr(Samp) ε Ansgnfnct$\big($1ingr(C1),Samp,C1$\big)$

E26.1 $[A]{:}A\ \varepsilon$ "$\underset{2}{\phi}(p)\underset{2}{(qr)}$"$.\supset.$1ingr$(A)\ \varepsilon \sim\Big($Ansgnfnct$\big($1ingr(C1), A,C1$\big)\Big)$

E26.2 $[A]{:}A\ \varepsilon$ "$\underset{2}{\phi}(p)\underset{2}{(qr)}$"$.\supset.$1ingr(C1) $\varepsilon \sim\Big($Ansgnfnct$\big($1ingr(A),C1, $A\big)\Big)$

E26.3 $[A]{:}A\ \varepsilon$ "$\dashv(p)$"$.\supset.$1ingr$(A)\ \varepsilon \sim\Big($Ansgnfnct$\big($1ingr(C1), A,C1$\big)\Big)$

E27 $[ABCD]\therefore A\ \varepsilon$ An(B,C,D)[23]$.\equiv{:}^1A\ \varepsilon$ Anarg$(B,C,D).\vee.A\ \varepsilon$ Ansgnfnct(B,C,D)

E27.0 1ingr(C1) ε An$\big($1ingr(Samp),C1,Samp$\big)$

E27.1 1ingr(C1) $\varepsilon \sim\Big($An$\big($4ingr(C1),C1,C1$\big)\Big)$

E28 $[AB]{:}A\ \varepsilon$ Arg1(B)[24]$.\equiv.^1[\exists C].C\ \varepsilon$ in(C1)$.A\ \varepsilon$ Anarg$\big($3ingr(C1), $B, C\big)$

E28.0 $[A]{:}A\ \varepsilon$ "(pq)"$.\supset.$2ingr$(A)\ \varepsilon$ Arg1(A)

E28.1 3ingr(C1) $\varepsilon \sim\big($Arg1(C1)$\big)$

E29 $[AB]{:}A\ \varepsilon$ Arg2(B)[25]$.\equiv.^1[\exists C].C\ \varepsilon$ in(C1)$.A\ \varepsilon$ Anarg$\big($4ingr(C1), $B, C\big)$

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[22] A is the functor analogous to B in C and D respectively.

[23] A is the component corresponding to B in C and D respectively.

[24] A is the first of two propositional arguments of B.

[25] A is the second of two propositional arguments of B.

E29.0 $[A]{:}A \mathbin{\varepsilon}$ "(pq)"$.{\supset}.3\mathrm{ingr}(A) \mathbin{\varepsilon} \mathrm{Arg2}(A)$

E29.1 $3\mathrm{ingr}(\mathrm{C1}) \mathbin{\varepsilon} \sim\Big(\mathrm{Arg2}\Big(\mathrm{Cmpl}\Big(\mathrm{vrb} \cap \mathrm{in}(\mathrm{C1}) \cap \mathrm{scd}\big(1\mathrm{ingr}(\mathrm{C1})\big)\Big)\Big)\Big)$

E30 $[AB]{.\cdot.}A \mathbin{\varepsilon} \mathrm{Eqvl1}(B)$[26]$.{\equiv}{:}^1\mathrm{Sgnfnct}(B) \mathbin{\varepsilon} \mathrm{cnf}\big(1\mathrm{ingr}(\mathrm{C1})\big){:}$

$^2[\exists C].C \mathbin{\varepsilon} \mathrm{prntm}(B).A \mathbin{\varepsilon} \mathrm{Arg1}(C)$

E30.0 $3\mathrm{ingr}(\mathrm{C1}) \mathbin{\varepsilon} \mathrm{Eqvl1}(\mathrm{C1})$

E30.1 $[A]{:}A \mathbin{\varepsilon}$ "ϕ(pp)"$.{\supset}.3\mathrm{ingr}(A) \mathbin{\varepsilon} \sim\big(\mathrm{Eqvl1}(A)\big)$

E30.2 $[A]{:}A \mathbin{\varepsilon}$ "ϕ(p)"$.{\supset}.3\mathrm{ingr}(A) \mathbin{\varepsilon} \sim\big(\mathrm{Eqvl1}(A)\big)$

E31 $[AB]{.\cdot.}A \mathbin{\varepsilon} \mathrm{Eqvl2}(B)$[27]$.{\equiv}{:}^1\mathrm{Sgnfnct}(B) \mathbin{\varepsilon} \mathrm{cnf}\big(1\mathrm{ingr}(\mathrm{C1})\big){:}$

$^2[\exists C].C \mathbin{\varepsilon} \mathrm{prntm}(B).A \mathbin{\varepsilon} \mathrm{Arg2}(C)$

E31.0 $4\mathrm{ingr}(\mathrm{C1}) \mathbin{\varepsilon} \mathrm{Eqvl2}(\mathrm{C1})$

E31.1 $[A]{:}A \mathbin{\varepsilon}$ "ϕ(pp)"$.{\supset}.4\mathrm{ingr}(A) \mathbin{\varepsilon} \sim\big(\mathrm{Eqvl2}(A)\big)$

E31.2 $3\mathrm{ingr}(\mathrm{C1}) \mathbin{\varepsilon} \sim\big(\mathrm{Eqvl2}(\mathrm{C1})\big)$

Most of the remaining explanations define terms which describe different objects as the system develops in the course of time. For example, a new definition extends the concept of 'meaningful expression', in the sense that expressions containing the new term are not meaningful before the definition is added to the system, but some of them are meaningful afterwards. All terms defined in this way end with the suffix 'p', which indicates that an object is or is not a such-and-such relative to some thesis of protothetic, which is specified by one of the arguments of the defined function. The analogous terms defined for stating the directives of ontology and mereology end with the suffixes 'o' and 'm' respectively[28].

E32 $[AB]{.\cdot.}A \mathbin{\varepsilon} \mathrm{thp}(B)$[29]$.{\equiv}{:}^1A \mathbin{\varepsilon} \mathrm{thp}.$

$^2B \mathbin{\varepsilon} \mathrm{thp}{:}$

$^3A \mathbin{\varepsilon} \mathrm{prcd}(B).{\vee}.A \mathbin{\varepsilon} \mathrm{Id}(B)$

E32.0 $\mathrm{C1} \mathbin{\varepsilon} \mathrm{thp}(\mathrm{C1})$

E32.1 $1\mathrm{ingr}(\mathrm{C1}) \mathbin{\varepsilon} \sim\big(\mathrm{thp}(\mathrm{C2})\big)$

E32.2 $\mathrm{C1} \mathbin{\varepsilon} \sim\Big(\mathrm{thp}\big(1\mathrm{ingr}(\mathrm{C2})\big)\Big)$

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26 A is the first argument of the equivalence B.

27 A is the second argument of the equivalence B.

28 Leśniewski29, pp. 68–9.

29 A is a thesis of this system of protothetic relative to B.

E32.3 $\text{C2}\ \varepsilon\ {\sim}\big(\text{thp(C1)}\big)$

E33 $[AB] \therefore A\ \varepsilon\ \text{frp}(B)$[30]$. \equiv :^1 A\ \varepsilon\ \text{thp}(B). \vee .[\exists CD].C\ \varepsilon\ \text{thp}(B).D\ \varepsilon\ \text{in}(C).A\ \varepsilon\ \text{Arg1}(D). \vee .[\exists CD].C\ \varepsilon\ \text{thp}(B).D\ \varepsilon\ \text{in}(C).A\ \varepsilon\ \text{Arg2}(D). \vee .[\exists CD].C\ \varepsilon\ \text{thp}(B).D\ \varepsilon\ \text{sbqntf}.D\ \varepsilon\ \text{in}(C).A\ \varepsilon\ \text{Cmpl}\big(\text{int}(D)\big)$

E33.0 $\text{4ingr(C1)}\ \varepsilon\ \text{frp(C1)}$

E33.1 $\text{Samp}\ \varepsilon\ {\sim}\big(\text{frp(C1)}\big)$

E34 $[ABC] \therefore A\ \varepsilon\ \text{1homosemp}(B,C)$[31]$. \equiv :^1 A\ \varepsilon\ \text{frp}(C).B\ \varepsilon\ \text{frp}(C). \vee .[\exists DE].D\ \varepsilon\ \text{thp}(C).E\ \varepsilon\ \text{in}(D).A\ \varepsilon\ \text{cnvar}(B,E). \vee .[\exists DEFG].D\ \varepsilon\ \text{thp}(C).E\ \varepsilon\ \text{in}(D).F\ \varepsilon\ \text{thp}(C).G\ \varepsilon\ \text{in}(F).A\ \varepsilon\ \text{An}(B,E,G)$

E34.0 $\text{3ingr(C1)}\ \varepsilon\ \text{1homosemp}\big(\text{4ingr(C1),C1}\big)$

E34.1 $\text{1ingr(C1)}\ \varepsilon\ {\sim}\Big(\text{1homosemp}\big(\text{4ingr(C1),C1}\big)\Big)$

The previous explanation covers those situations in which we can determine directly that two expressions belong to the same semantic category relative to some thesis. The following explanation, loosely speaking, says that the semantic category of an expression consists of all those expressions which belong to any domain which contains B and which is closed with respect to direct determination of belonging to the same semantic category. In this explanation there are two logically superfluous clauses: '$A\ \varepsilon\ \text{1homosemp}(A,C)$', and '$[D]:D\ \varepsilon\ a. \supset .D\ \varepsilon\ \text{1homosemp}(D,C)$'. The former clause simply emphasises that expressions which satisfy the following explanation also satisfy the previous one, while the other clause assures us that we can restrict our search to expressions in theses which are thp(C).

E35 $[ABC] :: A\ \varepsilon\ \text{homosemp}(B,C)$[32]$. \equiv :\because^1 A\ \varepsilon\ \text{1homosemp}(A,C).B\ \varepsilon\ \text{1homosemp}(B,C) :\because$ $^2[a] :: [D]:D\ \varepsilon\ a. \supset .D\ \varepsilon\ \text{1homosemp}(D,C) \therefore [DE] :D\ \varepsilon\ a.E\ \varepsilon\ \text{1homosemp}(D,C). \supset .E\ \varepsilon\ a \therefore B\ \varepsilon\ a \therefore \supset .A\ \varepsilon\ a$

E35.0 $\text{3ingr(C1)}\ \varepsilon\ \text{homosemp(C1,C1)}$

E35.1 $\text{C1}\ \varepsilon\ {\sim}\Big(\text{homosemp}\big(\text{2ingr(C1),C1}\big)\Big)$

E35.2 $\text{C1}\ \varepsilon\ {\sim}\Big(\text{homosemp}\big(\text{1ingr(C1),C1}\big)\Big)$

The following explanation describes a term A in an expression C which, relative to a thesis B, plays a suitable rôle as a constant. It does so because it is equiform with a constant D in some thesis relative to B, and because it is the 'structural analogue' of some expression E in the same semantic category as D. To prove that requirement 1 is

[30] A is an evident sentence relative to B.

[31] A is an expression which by direct evidence belongs to the same semantic category as B, relative to C.

[32] A is an expression belonging to the same semantic category as B relative to C.

independent of the other conditions, we must assume that the system of protothetic contains two equiform constants in different semantic categories, which is perfectly in accordance with the directives, but which we have in fact chosen not to do in our system *C*.

E36 $[ABCDE]{::}A\ \varepsilon\ \mathrm{constp}(B,C,D,E)^{33}.{\equiv}{\therefore}{}^{1}D\ \varepsilon\ \mathrm{homosemp}(E,B){\therefore}$

$^{2}[FG]{:}G\ \varepsilon\ \mathrm{thp}(B).F\ \varepsilon\ \mathrm{in}(G).{\supset}.D\ \varepsilon\ {\sim}\big(\mathrm{cnvar}(D,F)\big){\therefore}$

$^{3}A\ \varepsilon\ \mathrm{cnf}(D){:}$

$^{4}[\exists FGH].F\ \varepsilon\ \mathrm{in}(C).G\ \varepsilon\ \mathrm{thp}(B).H\ \varepsilon\ \mathrm{in}(G).A\ \varepsilon\ \mathrm{An}(E,F,H)$

E36.0 $1\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{constp}\big(\mathrm{C1},\mathrm{Samp},1\mathrm{ingr}(\mathrm{C1}),1\mathrm{ingr}(\mathrm{C1})\big)$

E36.1 $[A]{:}A\ \varepsilon\ \mathrm{thp}^{34}.A\ \varepsilon\ \text{“}\phi\big(\phi(\Lambda\Lambda)\phi\big)\text{”}.{\supset}.1\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ {\sim}\Big(\mathrm{constp}\big(A,\mathrm{Samp},8\mathrm{ingr}(A),1\mathrm{ingr}(A)\big)\Big)$

E36.2 $[A]{:}A\ \varepsilon\ \text{“}\phi(pp)\text{”}.{\supset}.3\mathrm{ingr}(A)\ \varepsilon\ {\sim}\Big(\mathrm{constp}\big(\mathrm{C25},A,11\mathrm{ingr}(\mathrm{C25}),3\mathrm{ingr}(\mathrm{C1})\big)\Big)$

E36.3 $[A]{:}A\ \varepsilon\ \text{“}\phi\text{-}(\Lambda\Lambda)\text{”}.{\supset}.1\mathrm{ingr}(A)\ \varepsilon\ {\sim}\Big(\mathrm{constp}\big(\mathrm{C1},A,1\mathrm{ingr}(\mathrm{C1}),1\mathrm{ingr}(\mathrm{C1})\big)\Big)$

E36.4 $3\mathrm{ingr}(\mathrm{C1})\ \varepsilon\ {\sim}\Big(\mathrm{constp}\big(\mathrm{C1},\mathrm{C1},3\mathrm{ingr}(\mathrm{C1}),4\mathrm{ingr}(\mathrm{C1})\big)\Big)$

E37 $[ABC]{:}A\ \varepsilon\ \mathrm{constp}(B,C).{\equiv}.{}^{1}[\exists DE].A\ \varepsilon\ \mathrm{constp}(B,C,D,E)$

E37.0 $1\mathrm{ingr}(\mathrm{Samp})\ \varepsilon\ \mathrm{constp}(\mathrm{C1},\mathrm{Samp})$

E37.1 $[A]{:}A\ \varepsilon\ \text{“}\Lambda(\phi\phi)\text{”}.{\supset}.1\mathrm{ingr}(A)\ \varepsilon\ {\sim}\big(\mathrm{constp}(\mathrm{C1},A)\big)$

The following explanation describes two expressions A and B in a larger expression D; A and B are able to belong to the same semantic category because of their respective analogues E and F, which do belong to the same semantic category relative to thesis C.

E38 $[ABCDEF]{\therefore}A\ \varepsilon\ \mathrm{quasihomosemp}(B,C,D,E,F)^{35}.{\equiv}{:}{}^{1}E\ \varepsilon\ \mathrm{homosemp}(F,C){:}$

$^{2}[\exists GHI].G\ \varepsilon\ \mathrm{in}(D).H\ \varepsilon\ \mathrm{thp}(C).I\ \varepsilon\ \mathrm{in}(H).A\ \varepsilon\ \mathrm{An}(E,G,I){:}$

$^{3}[\exists GHI].G\ \varepsilon\ \mathrm{in}(D).H\ \varepsilon\ \mathrm{thp}(C).I\ \varepsilon\ \mathrm{in}(H).B\ \varepsilon\ \mathrm{An}(F,G,I)$

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[33] A is suited to be a constant, relative to B, in C, by means of D and E.

[34] This example assumes that thesis A has been added to system C immediately after thesis *C4*, and instead of *C5*. In this, as in all examples of possible theses below, there is a legitimate way either to extend system C to include the proposed thesis, or to construct an alternative system which includes it.

[35] A is suited to belong to the same semantic category as B, relative to C, in D, by means of E and F.

E38.0 $[A]{:}A\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner \phi(pp) \urcorner$"$.\supset.$7ingr$(A)\ \varepsilon$ quasihomosemp$\big($8ingr(A),C1$,A,$3ingr(C1),4ingr(C1)$\big)$

E38.1 $[A]{:}A\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner \phi(pp) \urcorner$"$.\supset.$7ingr$(A)\ \varepsilon\ \sim\Big($quasihomosemp$\big($5ingr$(A)$,C1$,A,$3ingr (C1), 1ingr(C1)$\big)\Big)$

E38.2 $[A]{:}A\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner \phi(pp) \urcorner$"$.\supset.$7ingr$(A)\ \varepsilon\ \sim\Big($quasihomosemp$\big($8ingr$(A)$,C1$,A,$4ingr (C1), 4ingr(C1)$\big)\Big)$

E38.3 $[A]{:}A\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner \phi(pp) \urcorner$"$.\supset.$7ingr$(A)\ \varepsilon\ \sim\Big($quasihomosemp$\big($8ingr$(A)$,C1$,A,$3ingr (C1), 3ingr(C1)$\big)\Big)$

The following explanation says that a function A in expression C is suited to be an argument or functor in another function in C, relative to thesis B.

E39 $[ABCDE]{\therefore}A\ \varepsilon$ fnctp(B,C,D,E)[36]$.\equiv{:}^1D\ \varepsilon$ homosemp(E,B).

$^2A\ \varepsilon$ genfnct(D):

$^3[\exists FGH].F\ \varepsilon$ in$(C).G\ \varepsilon$ thp$(B).H\ \varepsilon$ in$(G).A\ \varepsilon$ An(E,F,H)

E39.0 $[A]{:}A\ \varepsilon$ "$\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\phi(qp)\big) \urcorner$"$.\supset.$Eqvl2$\big($Essnt$(A)\big)\ \varepsilon$ fnctp$\big($C1$,A,$C1, 4ingr(C1)$\big)$

E39.1 $[A]{:}A\ \varepsilon$ "$\phi(pp)(pp)$"$.\supset.$Sgnfnct$(A)\ \varepsilon\ \sim\Big($fnctp$\big($C1$,A,$C1, 1ingr(C1)$\big)\Big)$

E39.2 $[A]{:}A\ \varepsilon$ "$\phi\big(p\phi(p)\big)$"$.\supset.$Eqvl2$(A)\ \varepsilon\ \sim\Big($fnctp$\big($C1$,A,$C1, 4ingr(C1)$\big)\Big)$

E39.3 $[A]{:}A\ \varepsilon$ "$\phi\big(p\phi(pp)p\big)$"$.\supset.$Cmpl$\Big($vrb $\cap$ in(A) $\cap$ scd$\big($3ingr$(A)\big)$ $\cap$ prcd$\big($9ingr$(A$ $)\big)\Big)\ \varepsilon\ \sim\Big($fnctp$\big($C1$,A,$C1, 4ingr(C1)$\big)\Big)$

The following explanations describe conditions which are useful for describing the bracketed expressions which follow the defined terms in definitions.

E40 $[ABCDEF]{\therefore}A\ \varepsilon$ varp$(B,C,D,E,F).\equiv{:}^1E\ \varepsilon$ homosemp(B,C):

$^2[\exists GHI].G\ \varepsilon$ in$(D).H\ \varepsilon$ thp$(C).I\ \varepsilon$ in(H) $.F\ \varepsilon$ An(E,G,I):

$^3F\ \varepsilon$ in$\Big($Eqvl1$\big($Essnt$(D)\big)\Big)$.

$^4A\ \varepsilon$ cnvar(F,D)

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[36] A is a function in an appropriate context relative to B, in C, by means of D and E.

E40.0 $10\mathrm{ingr}(\mathrm{C29})\ \varepsilon\ \mathrm{varp}\Big(3\mathrm{ingr}(\mathrm{C1}),\mathrm{C1},\mathrm{C29},3\mathrm{ingr}(\mathrm{C1}),7\mathrm{ingr}(\mathrm{C29})\Big)$

E40.1 $10\mathrm{ingr}(\mathrm{C29})\ \varepsilon\ \sim\Big(\mathrm{varp}\Big(1\mathrm{ingr}(\mathrm{C1}),\mathrm{C1},\mathrm{C29},3\mathrm{ingr}(\mathrm{C1}),7\mathrm{ingr}(\mathrm{C29})\Big)\Big)$

E40.2 $10\mathrm{ingr}(\mathrm{C29})\ \varepsilon\ \sim\Big(\mathrm{varp}\Big(3\mathrm{ingr}(\mathrm{C1}),\mathrm{C1},\mathrm{C29},4\mathrm{ingr}(\mathrm{C1}),7\mathrm{ingr}(\mathrm{C29})\Big)\Big)$

E40.3 $4\mathrm{ingr}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C95})\Big)\Big)\ \varepsilon\ \sim\Big(\mathrm{varp}\Big(4\mathrm{ingr}(\mathrm{C1}),\mathrm{C1},\mathrm{C95},4\mathrm{ingr}(\mathrm{C1}),4\mathrm{ingr}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C95})\Big)\Big)\Big)\Big)$

E40.4 $4\mathrm{ingr}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C95})\Big)\Big)\ \varepsilon\ \sim\Big(\mathrm{varp}\Big(4\mathrm{ingr}(\mathrm{C1}),\mathrm{C1},\mathrm{C95},4\mathrm{ingr}(\mathrm{C1}),\mathrm{Eqvl2}\Big(\mathrm{Eqvl1}\Big(\mathrm{Essnt}(\mathrm{C95})\Big)\Big)\Big)\Big)$

E41 $[ABCDE]::A\ \varepsilon\ \mathrm{prntmp}(B,C,D,E).\equiv.\therefore{}^1D\ \varepsilon\ \mathrm{homosemp}(B,B).$
${}^2E\ \varepsilon\ \mathrm{prntm}(D).$
${}^3A\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(C)\Big)\Big).$
${}^4\mathrm{arg}(A)\ \infty\ \mathrm{arg}(E).\therefore$
${}^5[FG]:F\ \varepsilon\ \mathrm{arg}(A).G\ \varepsilon\ \mathrm{arg}(E).\Big(\mathrm{arg}(A)\cap \mathrm{prcd}(F)\Big)\ \infty\ \Big(\mathrm{arg}(E)\cap\mathrm{prcd}(G)\Big).\supset.[\exists HI].F\ \varepsilon\ \mathrm{varp}(G,B,C,H,I)$

E41.0 $\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C43})\Big)\Big)\Big)\ \varepsilon\ \mathrm{prntmp}\Big(\mathrm{C29},\mathrm{C43},\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C29})\Big),\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C29})\Big)\Big)\Big)\Big)$

E41.1 $[AB]:A\ \varepsilon\ \mathrm{thp}^{37}.A\ \varepsilon\ \text{“}\llcorner pqr\lrcorner\ulcorner\phi\Big(\phi\Big(\phi(pq)r\Big)\phi\underset{1}{(pq)}\underset{1}{(r)}\Big)\urcorner\text{”}.B\ \varepsilon\ \text{“}\llcorner pq\lrcorner\ulcorner\phi\Big(\phi(pq)\phi(pq)\Big)\urcorner\text{”}.\supset.\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(B)\Big)\Big)\Big)\ \varepsilon\ \sim\Big(\mathrm{prntmp}\Big(A,B,\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big),\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\Big)\Big)\Big)\Big)$

E41.2 $\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C43})\Big)\Big)\Big)\ \varepsilon\ \sim\Big(\mathrm{prntmp}\Big(\mathrm{C29},\mathrm{C43},\mathrm{C1},\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(\mathrm{C29})\Big)\Big)\Big)\Big)\Big)$

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[37] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

E41.3 $\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl1}\big(\mathrm{Essnt(C43)}\big)\Big)\Big)\ \varepsilon \sim\Big(\mathrm{prntmp}\Big(\mathrm{C1,C43,C1,Cmpl}\big(\mathrm{prntm(C1}$
$\mathrm{))}\Big)\Big)$

E41.4 $\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt(C43)}\big)\Big)\Big)\ \varepsilon \sim\Big(\mathrm{prntmp}\Big(\mathrm{C1,C43,C1,Cmpl}\big(\mathrm{prntm(C1}$
$\mathrm{))}\Big)\Big)$

E41.5 $[A]{:}A\ \varepsilon$ “$\llcorner fg\lrcorner\ulcorner\phi\big(\llcorner p\lrcorner\ulcorner\phi\big(f(p)g(p)\big)\urcorner\phi{<}fg{>}\big)\urcorner$”$.\supset.\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)$
$\Big)\Big)\ \varepsilon \sim\Big(\mathrm{prntmp}\Big(\mathrm{C1},A,\mathrm{C1,Cmpl}\big(\mathrm{prntm(C1)}\big)\Big)\Big)$

E42 $[ABCDE]{:}A\ \varepsilon\ \mathrm{1prntmp}(B,C,D,E).\equiv.{}^{1}A\ \varepsilon\ \mathrm{prntmp}(B,C,D,E).$
$^{2}\mathrm{Uingr}(D)\ \varepsilon\ \mathrm{in}(E)$

E42.0 $\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt(C43)}\big)\Big)\Big)\ \varepsilon\ \mathrm{1prntmp}\Bigg(\mathrm{C29,C43,Eqvl2}\big(\mathrm{Essnt(C29)}\big),$
$\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt(C29)}\big)\Big)\Big)\Bigg)$

E42.1 $\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt(C43)}\big)\Big)\Big)\ \varepsilon \sim\Big(\mathrm{1prntmp}\Big(\mathrm{C1,C43,C1,Cmpl}\big(\mathrm{prntm(}$
$\mathrm{C1)}\big)\Big)\Big)$

E42.2 $[AB]{:}A\ \varepsilon\ \mathrm{thp}$[38]$.A\ \varepsilon$ “$\llcorner pq\lrcorner\ulcorner\phi\big(\phi(pq)\phi\underset{1}{(p)}\underset{1}{(q)}\big)\urcorner$”$.B\ \varepsilon$ “$\llcorner p\lrcorner\ulcorner\phi\big(p\phi(p)\big)\urcorner$”$.\supset.\mathrm{Cmpl}\Big($
$\mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(B)\big)\Big)\Big)\ \varepsilon \sim\Bigg(\mathrm{1prntmp}\Bigg(A,\ B,\ \mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big),\ \mathrm{Cmpl}\Bigg($
$\mathrm{prntm}\Big(\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big)\Big)\Bigg)\Bigg)\Bigg)$

E43 $[ABCDEFG]{:}A\ \varepsilon\ \mathrm{2prntmp}(B,C,D,E,F,G).\equiv.{}^{1}A\ \varepsilon\ \mathrm{prntmp}(B,C,D,E).$
$^{2}F\ \varepsilon\ \mathrm{prntm}(D).$
$^{3}\mathrm{Uprcd}(F)\ \varepsilon\ \mathrm{in}(E).$
$^{4}G\ \varepsilon\ \mathrm{simprntm}(F)$

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[38] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

E43.0 $[AB]\colon A\ \varepsilon$ thp[39] $.A\ \varepsilon\ “\llcorner pq\lrcorner\ulcorner\varphi\big(\varphi(pq)\varphi\underset{1}{(}p\underset{1}{)}(q)\big)\urcorner”.B\ \varepsilon\ “\llcorner p\lrcorner\ulcorner\varphi\big(\varphi(pp)\phi(p)\big)\urcorner”.\supset.\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(B)\Big)\Big)\Big)\ \varepsilon\ \mathrm{2prntmp}\Bigg(A, B, \mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big), \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Bigg(\mathrm{Sgnfnct}\ \Big(\mathrm{Eqvl2}\ \Big(\mathrm{Essnt}\ (A)\Big)\Big)\Bigg)\Bigg), \mathrm{Cmpl}\ \Bigg(\mathrm{vrb}\ \cap\ \mathrm{in}\ \Big(\mathrm{Eqvl2}\ \Big(\mathrm{Essnt}\ (A)\Big)\Big)\ \cap\ \mathrm{scd}\Bigg(\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\Bigg)\Bigg), \mathrm{Cmpl}\Bigg(\mathrm{vrb}\ \cap\ \mathrm{in}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\ \cap\ \mathrm{scd}\Bigg(\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\Bigg)\Bigg)\Bigg)$

E43.1 $\mathrm{C1}\ \varepsilon\ \sim\Bigg(\mathrm{2prntmp}\Bigg(\mathrm{C1},\ \mathrm{C1},\ \mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big),\ \mathrm{1ingr}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big)\Big), \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big)\Big)\Bigg), \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big)\Big)\Bigg)\Bigg)\Bigg)$

E43.2 $\mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C95)}\Big)\Big)\Bigg)\ \varepsilon\ \sim\Bigg(\mathrm{2prntmp}\Bigg(\mathrm{C8},\mathrm{C95},\mathrm{C8},\mathrm{Cmpl}\Big(\mathrm{prntm}(\mathrm{C8})\Big),\mathrm{Cmpl}\Big(\mathrm{prntm}\Big(\mathrm{Eqvl1(C8)}\Big)\Big),\mathrm{Cmpl}\Big(\mathrm{prntm(C8)}\Big)\Bigg)\Bigg)$

E43.3 $\mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C43)}\Big)\Big)\Bigg)\ \varepsilon\ \sim\Bigg(\mathrm{2prntmp}\Bigg(\mathrm{C29},\mathrm{C43},\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big),\ \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big)\Big)\Bigg),\ \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big)\Big)\Bigg), \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt(C29)}\Big)\Big)\Bigg)\Bigg)\Bigg)$

E43.4 $[AB]\colon A\ \varepsilon$ thp[40] $.A\ \varepsilon\ “\llcorner pq\lrcorner\ulcorner\varphi\big(\varphi(pq)\varphi\underset{1}{(}p\underset{1}{)}(q)\big)\urcorner”.B\ \varepsilon\ “\llcorner p\lrcorner\ulcorner\varphi\big(\varphi(pp)\phi(p)\big)\urcorner”.\supset. \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(B)\Big)\Big)\Bigg)\ \varepsilon\ \sim\Bigg(\mathrm{2prntmp}\Bigg(A, B, \mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big), \mathrm{Cmpl}\Bigg(\mathrm{prntm}\Bigg(\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\Bigg)\Bigg), \mathrm{Cmpl}\Bigg(\mathrm{vrb}\ \cap\ \mathrm{in}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\ \cap\ \mathrm{scd}\Bigg(\mathrm{Sgnfnct}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\Bigg)\Bigg), \mathrm{Cmpl}\Big(\mathrm{prntm(C1)}\Big)\Bigg)\Bigg)$

[39] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

[40] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

6.2. Meaningful expressions

There is no explanation of 'meaningful expression' among the official terminological explanations of $\mathfrak{S}_5$. However, Leśniewski published such an explanation in a footnote to his article on definitions, which does not generally concern protothetic[41]. The conditions of this explanation obtain in computative protothetic, but we have in our system two additional restrictions: that no variable in a meaningful expression may have the shape of a constant, and that no meaningful expression may contain a variable in some category unless there exists in the system a thesis containing a variable in that category. This last requirement is complex enough to require an explanation of its own:

E44 $[ABCDEFGH] \therefore A\ \varepsilon\ \mathrm{extvarp}(B, C, D, E, F, G, H)$[42] $. \equiv :{}^1 E\ \varepsilon\ \mathrm{thp}(D).$
$^2 F\ \varepsilon\ \mathrm{in}(E).$
$^3 A\ \varepsilon\ \mathrm{cnvar}(A, F).$
$^4 G\ \varepsilon\ \mathrm{homosemp}(A, D).$
$^5 H\ \varepsilon\ \mathrm{in}(C):$
$^6 H\ \varepsilon\ \mathrm{gnrl}. B\ \varepsilon\ \mathrm{Essnt}(H). G\ \varepsilon\ \mathrm{frp}(D). \vee .[\exists IJ]. I\ \varepsilon\ \mathrm{thp}(D). J\ \varepsilon\ \mathrm{in}(I). B\ \varepsilon\ \mathrm{An}(G, H, J)$

E44.0 $\mathrm{3ingr}\Big(\mathrm{Eqvl1}\Big(\mathrm{Essnt(C25)}\Big)\Big)\ \varepsilon\ \mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C26}, \mathrm{C25}, \mathrm{C25}, \mathrm{C25}, \mathrm{3ingr}\Big(\mathrm{Eqvl1}\Big(\mathrm{Essnt(C25)}\Big)\Big), \mathrm{Eqvl1(C26)}\Big)$

E44.1 $\mathrm{7ingr(C27)}\ \varepsilon \sim\Big(\mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C26}, \mathrm{C27}, \mathrm{Eqvl1(C27)}, \mathrm{Eqvl1(C27)}, \mathrm{Eqvl1(C27)}, \mathrm{Eqvl1(C26)}\Big)\Big)$

E44.2 $\mathrm{7ingr(C27)}\ \varepsilon \sim\Big(\mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C26}, \mathrm{C27}, \mathrm{C1}, \mathrm{Eqvl1(C27)}, \mathrm{Eqvl1(C27)}, \mathrm{Eqvl1(C26)}\Big)\Big)$

E44.3 $\mathrm{3ingr(C1)}\ \varepsilon \sim\Big(\mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C26}, \mathrm{C1}, \mathrm{C1}, \mathrm{C1}, \mathrm{C1}, \mathrm{Eqvl1(C26)}\Big)\Big)$

E44.4 $\mathrm{1ingr}\Big(\mathrm{Eqvl1(C79)}\Big)\ \varepsilon \sim\Big(\mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C26}, \mathrm{C79}, \mathrm{C79}, \mathrm{C79}, \mathrm{C79}, \mathrm{Eqvl1(C26)}\Big)\Big)$

E44.5 $\mathrm{7ingr(C27)}\ \varepsilon \sim\Big(\mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C1}, \mathrm{C27}, \mathrm{C27}, \mathrm{Eqvl1(C27)}, \mathrm{C27}, \mathrm{Eqvl1(C26)}\Big)\Big)$

E44.6 $\mathrm{7ingr(C27)}\ \varepsilon \sim\Big(\mathrm{extvarp}\Big(\mathrm{7ingr(C26)}, \mathrm{C26}, \mathrm{C27}, \mathrm{C27}, \mathrm{Eqvl1(C27)}, \mathrm{C27}, \mathrm{C26}\Big)\Big)$

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[41] Leśniewski31, pp. 301–2.

[42] A is a variable already existing in the semantic category to which B is suited to belong, in C, relative to D, and by means of E, F, G, and H.

E44a $[AB]\mathrel{:\cdot:}A\ \varepsilon\ \mathrm{propp}(B)$[43]$.\equiv::^{1}B\ \varepsilon\ \mathrm{thp}::$

$^{2}[\exists C].\therefore C\ \varepsilon\ \mathrm{vrb}.C\ \varepsilon\ \mathrm{frp}(B).A\ \varepsilon\ \mathrm{cnf}(C).\therefore[DE]:D\ \varepsilon\ \mathrm{thp}(B).E\ \varepsilon\ \mathrm{in}(D).\supset.C\ \varepsilon\ \sim\big(\mathrm{cnvar}(C,E)\big).\therefore\vee.[\exists C].C\ \varepsilon\ \mathrm{frp}(B).A\ \varepsilon\ \mathrm{genfnct}(C).\vee.A\ \varepsilon\ \mathrm{gnrl}::$

$^{3}[C].\therefore C\ \varepsilon\ \mathrm{trm}.C\ \varepsilon\ \mathrm{in}(A).\supset:C\ \varepsilon\ \mathrm{Id}(A).\vee.[\exists D].D\ \varepsilon\ \mathrm{qntf}.D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{int}(D).\vee.[\exists DE].D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{var}(E,D).\vee.C\ \varepsilon\ \mathrm{constp}(B,A)::$

$^{4}[CD]:D\ \varepsilon\ \mathrm{qntf}.D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{int}(D).\supset.[\exists EF].E\ \varepsilon\ \mathrm{in}(A).F\ \varepsilon\ \mathrm{var}(C,E)::$

$^{5}[CDE].\therefore E\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{cnvar}(D,E).\supset:C\ \varepsilon\ \mathrm{Id}(D).\vee.[\exists FG].C\ \varepsilon\ \mathrm{quasihomosemp}(D,B,A,F,G)::$

$^{6}[C].\therefore C\ \varepsilon\ \mathrm{gnrl}.C\ \varepsilon\ \mathrm{in}(A).\supset:C\ \varepsilon\ \mathrm{Id}(A).\vee.[\exists DEFG].D\ \varepsilon\ \mathrm{thp}(B).E\ \varepsilon\ \mathrm{in}(D).F\ \varepsilon\ \mathrm{in}(A).G\ \varepsilon\ \mathrm{homosemp}(B,B).G\ \varepsilon\ \mathrm{Anarg}(C,E,F)::$

$^{7}[CD].\therefore C\ \varepsilon\ \mathrm{gnrl}.C\ \varepsilon\ \mathrm{in}(A).D\ \varepsilon\ \mathrm{Essnt}(C).\supset:D\ \varepsilon\ \mathrm{vrb}.\vee.[\exists E].E\ \varepsilon\ \mathrm{frp}(B).D\ \varepsilon\ \mathrm{genfnct}(E)::$

$^{8}[C].\therefore C\ \varepsilon\ \mathrm{fnct}.C\ \varepsilon\ \mathrm{in}(A).\supset:C\ \varepsilon\ \mathrm{Id}(A).\vee.[\exists D].D\ \varepsilon\ \mathrm{gnrl}.D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{Essnt}(D).\vee.[\exists DE].C\ \varepsilon\ \mathrm{fnctp}(B,A,D,E)::$

$^{9}[CDEF].\therefore C\ \varepsilon\ \mathrm{qntf}.C\ \varepsilon\ \mathrm{in}(A).D\ \varepsilon\ \mathrm{int}(C).E\ \varepsilon\ \mathrm{thp}(B).F\ \varepsilon\ \mathrm{in}(E).F\ \varepsilon\ \mathrm{cnf}(D).\supset:[\exists G].G\ \varepsilon\ \mathrm{qntf}.G\ \varepsilon\ \mathrm{in}(E).F\ \varepsilon\ \mathrm{int}(G).\vee.[\exists G].G\ \varepsilon\ \mathrm{in}(E).F\ \varepsilon\ \mathrm{cnvar}(F,G)::$

$^{10}[CD]:D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{cnvar}(C,D).\supset.[\exists EFGHI].E\ \varepsilon\ \mathrm{extvarp}(C,A,B,F,G,H,I)$

E44a.0 $\mathrm{C1}\ \varepsilon\ \mathrm{propp}(\mathrm{C1})$

E44a.1 $[A]:A\ \varepsilon\ \text{“}\llcorner u\lrcorner\ulcorner u\urcorner\text{”}.\supset.A\ \varepsilon\ \sim\big(\mathrm{propp}(\mathrm{C25})\big)$

E44a.2 $\mathrm{2ingr}(\mathrm{C1})\ \varepsilon\ \sim\big(\mathrm{propp}(\mathrm{C1})\big)$

E44a.3 $\mathrm{Essnt}(\mathrm{C25})\ \varepsilon\ \sim\big(\mathrm{propp}(\mathrm{C25})\big)$

E44a.4 $[A]:A\ \varepsilon\ \text{“}\llcorner p\lrcorner\ulcorner\phi(\Lambda\Lambda)\urcorner\text{”}.\supset.A\ \varepsilon\ \sim\big(\mathrm{propp}(\mathrm{C1})\big)$

E44a.5 $[A]:A\ \varepsilon\ \text{“}\llcorner p\lrcorner\ulcorner p(p)\urcorner\text{”}.\supset.A\ \varepsilon\ \sim\big(\mathrm{propp}(\mathrm{C79})\big)$

E44a.6 $[AB]:A\ \varepsilon\ \mathrm{thp}$[44]$.A\ \varepsilon\ \text{“}\llcorner f\lrcorner\ulcorner\phi\big(\llcorner p\lrcorner\ulcorner f(p)\urcorner\text{ϙ}\underset{1}{<}f\underset{1}{>}\big)\urcorner\text{”}.B\ \varepsilon\ \text{“ϙ}\underset{1}{<}f\underset{1}{>}\text{”}.\supset.B\ \varepsilon\ \sim\big(\mathrm{propp}(A)\big)$

[43] A is a meaningful expression relative to B.

[44] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

E44a.7 $[AB]\colon A\ \varepsilon\ \mathrm{thp}$[45]$.\, A\ \varepsilon\ \text{“}\llcorner pq\lrcorner \ulcorner \phi\big(\phi(pq)\phi\underset{1}{(}p\underset{1}{)}(q)\big)\urcorner\text{”}.\, B\ \varepsilon\ \text{“}\llcorner p\lrcorner \ulcorner \phi\underset{1}{(}p\underset{1}{)}\urcorner\text{”}.\supset.\, B\ \varepsilon\ \sim\big(\mathrm{propp}(A)\big)$

E44a.8 $[A]\colon A\ \varepsilon\ \text{“}\phi(\Lambda\Lambda)(\Lambda\Lambda)\text{”}.\supset.\, A\ \varepsilon \sim\big(\mathrm{propp}(\mathrm{C1})\big)$

E44a.9 $[A]\colon A\ \varepsilon\ \text{“}\llcorner \Lambda\lrcorner \ulcorner \phi(\Lambda\Lambda)\urcorner\text{”}.\supset.\, A\ \varepsilon \sim\big(\mathrm{propp}(\mathrm{C25})\big)$

E44a.10 $\mathrm{C26}\ \varepsilon \sim\big(\mathrm{propp}(\mathrm{C24})\big)$

As with Leśniewski's explanation of expressions meaningful in $\mathfrak{S}_5$, *E44a* is not actually used in the directives or in any later explanation, but the purpose of certain conditions in later explanations becomes clearer when they are compared with the conditions of *E44a*.

6.3. Explanations specific to system C

At this point we can state the final explanations which we require before stating the directives of system *C*.

E45 $[AB]\colon A\ \varepsilon\ \mathrm{Negt}(B)$[46]$.\equiv.\, {}^{1}A\ \varepsilon\ \mathrm{Eqvl1}(B).$

$$^{2}\mathrm{Eqvl2}(B)\ \varepsilon\ \mathrm{cnf}\big(4\mathrm{ingr}(\mathrm{C1})\big)$$

E45.0 $3\mathrm{ingr}(\mathrm{C1})\ \varepsilon\ \mathrm{Negt}(\mathrm{C1})$

E45.1 $\mathrm{C1}\ \varepsilon \sim\big(\mathrm{Negt}(\mathrm{C1})\big)$

E45.2 $3\mathrm{ingr}(\mathrm{C7})\ \varepsilon \sim\big(\mathrm{Negt}(\mathrm{C7})\big)$

E46 $[ABC]\colon\colon A\ \varepsilon\ \mathrm{primp}(B,C)$[47]$.\equiv\therefore {}^{1}A\ \varepsilon\ \mathrm{thp}(C).$

$^{2}B\ \varepsilon\ \mathrm{trm}.$

$^{3}B\ \varepsilon\ \mathrm{in}(A)\colon$

$^{4}[\exists D].\, D\ \varepsilon\ \mathrm{thp}.\, D\ \varepsilon\ \mathrm{prcd}(A)\therefore$

$^{5}[D]\colon D\ \varepsilon\ \mathrm{in}(A).\supset.\, B\ \varepsilon \sim\big(\mathrm{cnvar}(B,D)\big)\therefore$

$^{6}[DE]\colon D\ \varepsilon\ \mathrm{thp}(A).\, E\ \varepsilon\ \mathrm{in}(D).\, E\ \varepsilon\ \mathrm{cnf}(B).\, E\ \varepsilon\ \mathrm{homosemp}(B,C).\supset.\, D\ \varepsilon\ \mathrm{Id}(A)$

E46.0 $\mathrm{C5}\ \varepsilon\ \mathrm{primp}\big(\mathrm{Eqvl2}(\mathrm{C5}),\mathrm{C5}\big)$

E46.1 $\mathrm{Eqvl2}(\mathrm{C5})\ \varepsilon \sim\Big(\mathrm{primp}\big(\mathrm{Eqvl2}(\mathrm{C5}),\mathrm{C5}\big)\Big)$

[45] This example assumes that thesis *A* has been added to system *C* immediately after thesis *C99*.

[46] *A* is the expression negated in *B*.

[47] *A* is an introductory thesis for *B*, relative to *C*.

E46.2 $\quad$ C3 $\varepsilon \sim\Big(\text{primp}\big(\text{Eqvl1(C3),C3}\big)\Big)$

E46.3 $\quad$ C5 $\varepsilon \sim\big(\text{primp(C6,C5)}\big)$

E46.4 $\quad$ C1 $\varepsilon \sim\Big(\text{primp}\big(\text{Eqvl1(C1),C1}\big)\Big)$

E46.5 $\quad$ C25 $\varepsilon \sim\Bigg(\text{primp}\bigg(\text{3ingr}\Big(\text{Eqvl1}\big(\text{Essnt(C25)}\big)\Big)\text{,C25}\bigg)\Bigg)$

E46.6 $\quad$ C3 $\varepsilon \sim\Big(\text{primp}\big(\text{Eqvl2(C3)C3}\big)\Big)$

E47 $\quad [ABC]{:}{:}A\ \varepsilon\ \text{cruxp}(B,C)$[48]$.{\equiv}.{\therefore}^{1}B\ \varepsilon\ \text{thp}(C){:}$

$^{2}A\ \varepsilon\ \text{Id}(B).\vee.A\ \varepsilon\ \text{Negt}(B){:}$

$^{3}\text{1ingr}(A)\ \varepsilon\ \text{trm}.{\therefore}$

$^{4}[DE]{:}D\ \varepsilon\ \text{prntm}(A).E\ \varepsilon\ \text{arg}(D).{\supset}.E\ \varepsilon\ \text{trm}$

E47.0 $\quad$ C6 ε cruxp(C6,C6)

E47.1 $\quad$ $\text{Eqvl1}\big(\text{Eqvl1(C3)}\big)\ \varepsilon \sim\Big(\text{cruxp}\big(\text{Eqvl1(C3),C3}\big)\Big)$

E47.2 $\quad$ C1 $\varepsilon \sim\big(\text{cruxp(C2,C2)}\big)$

E47.3 $\quad$ Eqvl1(C26) $\varepsilon \sim\big(\text{cruxp(C26,C26)}\big)$

E47.4 $\quad$ Eqvl1(C3) $\varepsilon \sim\big(\text{cruxp(C3,C3)}\big)$

E48 $\quad [ABCDEFGH]{:}{:}A\ \varepsilon\ \text{rspcrxp}(B,C,D,E,F,G,H)$[49]$.{\equiv}.{\therefore}^{1}A\ \varepsilon\ \text{cruxp}(D,C).$

$^{2}B\ \varepsilon\ \text{cruxp}(E,C).$

$^{3}F\ \varepsilon\ \text{trm}.$

$^{4}G\ \varepsilon\ \text{homosemp}(H,C).$

$^{5}G\ \varepsilon\ \text{cnf}(H).$

$^{6}G\ \varepsilon\ \text{in}(A).$

$^{7}\big(\text{vrb} \cap \text{in}(A)\big) \propto \big(\text{vrb} \cap \text{in}(B)\big).$

$^{8}\big(\text{vrb} \cap \text{in}(A) \cap \text{prcd}(G)\big) \propto \big(\text{vrb} \cap \text{in}(B) \cap \text{prcd}(F)\big).{\therefore}$

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[48] A is a critical expression decided by B, relative to C.

[49] A is a critical expression corresponding to B, relative to C, in E and F respectively, by means of G and H.

$^{9}[IJ]{:}I\ \varepsilon\ \mathrm{vrb}.I\ \varepsilon\ \mathrm{in}(B).J\ \varepsilon$ $\mathrm{vrb}.J\ \varepsilon\ \mathrm{in}(A).I\ \varepsilon \sim\!\big(\mathrm{Id}(F)\big).\big(\mathrm{vrb}\cap \mathrm{in}(A)\cap \mathrm{prcd}(J)\big) \infty \big(\mathrm{vrb}\cap \mathrm{in}(B)\cap \mathrm{prcd}(I)\big).\supset.I$ $\varepsilon\ \mathrm{cnf}(J)$

E48.0 C6 ε rspcrxp$\big($Eqvl1(C1),C6,C6,C1,Eqvl1(C1),C6,C6$\big)$

E48.1 Eqvl2(C5) $\varepsilon \sim\!\Big($rspcrxp$\big($Eqvl1(C1),C6,C5,C1,Eqvl1(C1),Eqvl2(C5),C6$\big)\Big)$

E48.2 C6 $\varepsilon \sim\!\Big($rspcrxp$\big($Eqvl2(C1),C6,C6,C1,Eqvl2(C1),C6,C6$\big)\Big)$

E48.3 C31 $\varepsilon \sim\!\Big($rspcrxp$\Big($C31,C31,C31,C31,Cmpl$\big($3ingr(C31) $\cup$ 4ingr(C31)$\big)$, 3ingr(C31),C6$\Big)\Big)$

E48.4 $[A]{:}A\ \varepsilon$ thp[50]$.A\ \varepsilon$ "V(ΛΛ)"$.\supset.$C6 $\varepsilon \sim\!\Big($rspcrxp$\big($Eqvl1(C1), A,C6,C1,Eqvl1(C1),C6, 1ingr$(A)\big)\Big)$

E48.5 C6 $\varepsilon \sim\!\Big($rspcrxp$\big($Eqvl1(C1),C6,C6,C1,Eqvl1(C1),C6,Eqvl1(C1)$\big)\Big)$

E48.6 Eqvl1(C30) $\varepsilon \sim\!\Big($rspcrxp$\big($C44,C44,C30,C44, 1ingr(C44),C30,C30$\big)\Big)$

E48.7 C31 $\varepsilon \sim\!\Big($rspcrxp$\big($Eqvl1(C1),C31,C31,C1,Eqvl1(C1),C31,C31$\big)\Big)$

E48.8 C31 $\varepsilon \sim\!\Big($rspcrxp$\big($C31,C31,C31,C31, 1ingr(C31), 3ingr(C31),C6$\big)\Big)$

E48.9 C31 $\varepsilon \sim\!\Big($rspcrxp$\big($C44,C44,C31,C44, 1ingr(C44), 1ingr(C31), 1ingr(C31)$\big)\Big)$

E49 $[ABCDEF]{\therefore}A\ \varepsilon\ \mathrm{simcrxp}(B,C,D,E,F)$[51]$.\equiv{:}^{1}A\ \varepsilon\ \mathrm{rspcrxp}\big(B,\ C,\ D,\ E,\ 1\mathrm{ingr}(B),$ $1\mathrm{ingr}(A), 1\mathrm{ingr}(A)\big).$

$^{2}F\ \varepsilon\ \mathrm{cnf}\big(1\mathrm{ingr}(A)\big).$

$^{3}1\mathrm{ingr}(A)\ \varepsilon\ \mathrm{homosemp}(F,C){:}$

$^{4}A\ \varepsilon\ \mathrm{Id}(D).B\ \varepsilon\ \mathrm{Id}(E).\vee.A\ \varepsilon\ \mathrm{Negt}(D$ $).B\ \varepsilon\ \mathrm{Negt}(E)$

E49.0 C45 ε simcrxp$\big($C31,C45,C45,C31, 1ingr(C45)$\big)$

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[50] This example assumes that in constructing system C we have modified thesis *C95* and all subsequent theses, replacing any words equiform with 'ϕ' by means of words equiform with 'V'; thesis A in the resulting system corresponds to thesis *C96* in system C.

[51] A is a critical expression decided in the same way as B, relative to C, in D and E respectively, by means of F.

E49.1 $\mathrm{C44}\ \varepsilon \sim\Big(\mathrm{simcrxp}\Big(\mathrm{C31,C44,C44,C31}, \mathrm{1ingr(C44)}\Big)\Big)$

E49.2 $\mathrm{C45}\ \varepsilon \sim\Big(\mathrm{simcrxp}\Big(\mathrm{C31,C45,C45,C31}, \mathrm{1ingr(C31)}\Big)\Big)$

E49.3 $[A]{:}A\ \varepsilon\ \mathrm{thp}^{52}.A\ \varepsilon\ \text{“}\llcorner f \lrcorner \ulcorner \phi\big(\llcorner p \lrcorner \ulcorner f(p)\urcorner \vdash \underset{1}{<} f \underset{1}{>}\big)\urcorner\text{”}.\supset.\mathrm{C45}\ \varepsilon \sim\Bigg(\mathrm{simcrxp}\Bigg(\mathrm{C31}, A,$
$\mathrm{C45,C31}, \mathrm{1ingr}\Big(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(A)\Big)\Big)\Bigg)\Bigg)$

E49.4 $\mathrm{C44}\ \varepsilon \sim\Big(\mathrm{simcrxp}\Big(\mathrm{C30,C44,C44,C30}, \mathrm{1ingr(C44)}\Big)\Big)$

E50 $[ABCDE]{::}A\ \varepsilon\ \mathrm{dsccrxp}(B, C, D, E)^{53}.\equiv{.}\therefore{}^{1}A\ \varepsilon\ \mathrm{rspcrxp}\Big(B, C, D, E, \mathrm{1ingr}(B), \mathrm{1ingr}(A), \mathrm{1ingr}(A)\Big){:}$

${}^{2}A\ \varepsilon\ \mathrm{Id}(D).B\ \varepsilon\ \mathrm{Negt}(E).\vee.A\ \varepsilon\ \mathrm{Negt}(D).B\ \varepsilon\ \mathrm{Id}(E)\therefore$

${}^{3}[FG] : F\ \varepsilon\ \mathrm{cruxp}\ (G,\ C)\ .\ \mathrm{1ingr}\ (F)\ \varepsilon\ \mathrm{cnf}\ \Big(\mathrm{1ingr}(A)\Big).\mathrm{1ingr}(F)\ \varepsilon\ \mathrm{homosemp}(\mathrm{1ingr}(A), C\Big).F\ \varepsilon \sim\Big(\mathrm{cnf}(A)\Big).\supset.[\exists HI].H\ \varepsilon\ \mathrm{simcrxp}\Big(F, C, I, G, \mathrm{1ingr}(B)\Big)$

50.0 $\mathrm{C6}\ \varepsilon\ \mathrm{dsccrxp}\Big(\mathrm{3ingr(C1)},\mathrm{C6,C6,C1}\Big)$

50.1 $\mathrm{2ingr(C1)}\ \varepsilon \sim\Big(\mathrm{dsccrxp}\Big(\mathrm{3ingr(C1)},\mathrm{C1}, \mathrm{2ingr(C1)},\mathrm{C1}\Big)\Big)$

50.2 $\mathrm{C6}\ \varepsilon \sim\Big(\mathrm{dsccrxp(C6,C6,C6,C6)}\Big)$

50.3 $\mathrm{Eqvl1(C30)}\ \varepsilon \sim\Big(\mathrm{dsccrxp(C57,C57,C30,C57)}\Big)$

E51 $[ABCD]{::}A\ \varepsilon\ \mathrm{plenp}(B, C, D)^{54}.\equiv{.}\therefore{}^{1}C\ \varepsilon\ \mathrm{cruxp}(D, B).$

${}^{2}A\ \varepsilon\ \mathrm{cnf}\Big(\mathrm{1ingr}(C)\Big).$

${}^{3}A\ \varepsilon\ \mathrm{homosemp}\Big(\mathrm{1ingr}(C), B\Big)\therefore$

${}^{4}[EFGHIJ]{:}E\ \varepsilon\ \mathrm{cruxp}(F, B).G\ \varepsilon\ \mathrm{prntm}(E).H\ \varepsilon\ \mathrm{arg}(G).I\ \varepsilon\ \mathrm{thp}(B).J\ \varepsilon\ \mathrm{constp}(B, I).\mathrm{1ingr}(E)\ \varepsilon\ \mathrm{homosemp}(A, B).J\ \varepsilon\ \mathrm{homosemp}(H, B).\supset.[\exists KLM].K\ \varepsilon\ \mathrm{rspcrxp}(E, B, L, F, H, M, J)\therefore$

${}^{5}[EF]{:}E\ \varepsilon\ \mathrm{cruxp}(F, B).\mathrm{1ingr}(E)\ \varepsilon\ \mathrm{homosemp}(A, B).\supset.[\exists GH].G\ \varepsilon\ \mathrm{dsccrxp}(E, B, H, F)$

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[52] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

[53] A is the only critical expression distinguishing its first word from that of critical expression B, relative to C, by means of D and E.

[54] A is a constant in a semantic category which has been exhausted relative to B, by means of C and D.

51.0 C6 ε plenp(C6,C6,C6)

51.1 C6 $\varepsilon \sim\big(\text{plenp(C6,C6,C5)}\big)$

51.2 Eqvl1(C1) $\varepsilon \sim\big(\text{plenp(C6,C6,C6)}\big)$

51.3 $[A]{:}A \;\varepsilon\; \text{thp}$[55]$.A \;\varepsilon\;$ “$\llcorner f \lrcorner \ulcorner \phi\big(\llcorner p \lrcorner \ulcorner f(p)\urcorner \vdash\!\dashv \underset{1}{<} f \underset{1}{>}\big)\urcorner$”$.\supset.1\text{ingr}\big(\text{Eqvl2}(A)\big) \;\varepsilon \sim\big(\text{plenp}(A,\text{C44,C44})\big)$

51.4 Our system does not contain an example which verifies the independence of condition 4 of *E51* because system *C* is constructed observing the convention that the critical theses for a new constant should be introduced immediately after its definition. If, conforming completely to the directives, we were to abandon this convention, and we were to introduce theses *C43* and *C44* immediately after thesis *C30*, we could then demonstrate the independence of condition 4 by affirming (truly) that 1ingr(C44) $\varepsilon \sim\big(\text{plenp(C44,C44,C44)}\big)$.

51.5 1ingr(C45) $\varepsilon \sim\big(\text{plenp(C45,C45,C45)}\big)$

E52 $[ABCa]{::}A \;\varepsilon\; \text{psubst}(B,C,a)$[56]$.\equiv{::}{}^1A \;\varepsilon\; \text{Cmpl}(a).$

$^2B \;\varepsilon\; \text{expr}.$

$^3C \;\varepsilon\; \text{expr}.$

$^4B \;\varepsilon\; \text{in}(C).$

$^5a \propto \big(\text{vrb} \cap \text{in}(B)\big){\therefore}$

$^6[DE]{:}D \;\varepsilon\; \text{vrb}.D \;\varepsilon\; \text{in}(B).E \;\varepsilon\; a.E \;\varepsilon \sim\big(\text{cnf}(D)\big).$
$\big(a \cap \text{prcd}(E)\big) \propto \big(\text{vrb} \cap \text{in}(B) \cap \text{prcd}(D)\big).\supset.[\exists F].D \;\varepsilon\; \text{var}(F,C){\therefore}$

$^7[DEFG]{:}D \;\varepsilon\; \text{cnvar}(E,C).D \;\varepsilon\; \text{in}(B).E \;\varepsilon\; \text{in}(B)$
$.F \;\varepsilon\; a.G \;\varepsilon\; a.\big(a \cap \text{prcd}(F)\big) \propto \big(\text{vrb} \cap \text{in}(B) \cap \text{prcd}(D)\big).\big(a \cap \text{prcd}(G)\big) \propto \big(\text{vrb} \cap \text{in}(B) \cap \text{prcd}(E)\big).\supset.F \;\varepsilon\; \text{cnf}(G){::}$

$^8[DE]{\therefore}D \;\varepsilon\; \text{vrb}.D \;\varepsilon\; \text{in}(B).E \;\varepsilon\; a.\big(a \cap \text{prcd}(E)\big)$
$\propto \big(\text{vrb} \cap \text{in}(B) \cap \text{prcd}(D)\big).\supset{:}E \;\varepsilon\; \text{trm}.\vee.E \;\varepsilon\; \text{fnct}.\vee.E \;\varepsilon\; \text{gnrl}.\vee.E \;\varepsilon\; \text{cnf}(D)$

E52.0 C1 $\varepsilon\; \text{psubst}\big(\text{C1,C1,vrb} \cap \text{in(C1)}\big)$

E52.1 C2 $\varepsilon \sim\Big(\text{psubst}\big(\text{C1,C1,vrb} \cap \text{in(C1)}\big)\Big)$

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[55] This example assumes that thesis *A* has been added to system *C* immediately after thesis *C99*.

[56] *A* is a partial substitution of *B* from *C*, by means of *a*.

E52.2 $\mathrm{C1}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{Ccl}\Big(\mathrm{1ingr(C2)} \cup \mathrm{2ingr(C2)} \cup \mathrm{5ingr(C2)} \cup \mathrm{6ingr(C2)} \cup \mathrm{7ingr}$
$\mathrm{(C2)}\Big),\mathrm{C2},\mathrm{vrb} \cap \mathrm{in(C1)}\Big)\Big)$

E52.3 $\mathrm{C1}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{Eqvl1(C2)},\mathrm{Ccl}\Big(\mathrm{Eqvl1(C2)} \cup \mathrm{Uingr(C2)}\Big),\mathrm{vrb} \cap \mathrm{in(C1)}\Big)\Big)$

E52.4 $\mathrm{C1}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{C1},\mathrm{C2},\mathrm{vrb} \cap \mathrm{in(C1)}\Big)\Big)$

E52.5 $\mathrm{C1}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{Ccl}\Big(\mathrm{Eqvl1(C2)} \cup \mathrm{Eqvl2(C2)}\Big),\mathrm{C2},\mathrm{vrb} \cap \mathrm{in(C1)}\Big)\Big)$

E53.6 $\mathrm{C1}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{C7},\mathrm{C7},\mathrm{vrb} \cap \mathrm{in(C1)}\Big)\Big)$

E53.6 $\mathrm{Eqvl1(C8)}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{Eqvl1}\Big(\mathrm{Essnt(C43)}\Big),\mathrm{C43},\mathrm{vrb} \cap \mathrm{in}\Big(\mathrm{Eqvl1(C8)}\Big)\Big)\Big)$

E53.7 $\mathrm{2ingr(C26)}\ \varepsilon \sim\!\Big(\mathrm{psubst}\Big(\mathrm{Essnt}\Big(\mathrm{Eqvl1(C26)}\Big),\mathrm{Eqvl1(C26)}, \mathrm{2ingr(C26)}\Big)\Big)$

E54 $[ABCDEF]\!::\!A\ \varepsilon\ \mathrm{rspvarp}(B,C,D,E,F)$[57]$.\equiv.\therefore{}^{1}C\ \varepsilon\ \mathrm{thp}.$
${}^{2}A\ \varepsilon\ \mathrm{vrb} \cap \mathrm{in}(D)$[58]
${}^{3}F\ \varepsilon\ \mathrm{var}(B,E).$
${}^{4}D\ \varepsilon\ \mathrm{psubst}\Big(\mathrm{Essnt}(E), E,\mathrm{vrb} \cap \mathrm{in}(D$
$)\Big).$
${}^{5}\Big(\mathrm{vrb} \cap \mathrm{in}(D) \cap \mathrm{prcd}(A)\Big) \infty \Big(\mathrm{vrb} \cap$
$\mathrm{in}\Big(\mathrm{Essnt}(E)\Big) \cap \mathrm{prcd}(F)\Big).\therefore$
${}^{6}[GH]\!:\!G\ \varepsilon\ \mathrm{cnvar}(H,E).G\ \varepsilon \sim\!\Big(\mathrm{Id}(H$
$)\Big).\supset.[\exists IJ]G\ \varepsilon\ \mathrm{quasihomosemp}(H,C,E,I,J)$

E54.0 $\mathrm{C6}\ \varepsilon\ \mathrm{rspvarp}\Big(\mathrm{4ingr(C26)},\mathrm{C1},\mathrm{C6},\mathrm{Eqvl1(C26)}, \mathrm{7ingr(C26)}\Big)$

E54.1 $\mathrm{C6}\ \varepsilon \sim\!\Big(\mathrm{rspvarp}\Big(\mathrm{4ingr(C26)},\mathrm{Samp},\mathrm{C6},\mathrm{Eqvl1(C26)}, \mathrm{7ingr(C26)}\Big)\Big)$

E54.2 $[A]\!:\!A\ \varepsilon$ “$\llcorner f \lrcorner \ulcorner f(\Lambda\Lambda)\urcorner$”$.\supset.\mathrm{C1}\ \varepsilon \sim\!\Big(\mathrm{rspvarp}\Big(\mathrm{2ingr}(A),\mathrm{C1},\mathrm{C1}, A, \mathrm{5ingr}(A)\Big)\Big)$

E54.3 $\mathrm{C6}\ \varepsilon \sim\!\Big(\mathrm{rspvarp}\Big(\mathrm{C26},\mathrm{C1},\mathrm{C6},\mathrm{Eqvl1(C26)}, \mathrm{7ingr(C26)}\Big)\Big)$

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[57] A is a term corresponding to a word F bound by B, relative to C, in D and E respectively.

[58] The expression ‘in(D)’ can be omitted from this explanation without altering the meaning of ‘rspvarp’. I have nevertheless retained it because the proof relies on certain presuppositions which Leśniewski would probably have wished to avoid.

E54.4 $\text{1ingr(C1)}\ \varepsilon \sim\Big(\text{rspvarp}\Big(\text{4ingr(C26),C1,C1,Eqvl1(C26)}, \text{7ingr(C26)}\Big)\Big)$

E54.5 $[A]{:}A\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner \phi(pp)\urcorner$"$.\supset.\text{Eqvl2(C1)}\ \varepsilon \sim\Big(\text{rspvarp}\Big(\text{2ingr}(A)\text{,C1,C1,}\ A\text{,Eqvl1}\Big($

$\text{Essnt}(A)\Big)\Big)\Big)$

E54.6 $[AB]{:}A\ \varepsilon\ \text{thp}$[59]$.A\ \varepsilon$ "$\Lambda(\Lambda\Lambda)$"$.B\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner p(p\Lambda)\urcorner$"$.\supset.\text{1ingr}(A)\ \varepsilon \sim\Big(\text{rspvarp}\Big($

$\text{2ingr}(B), A, A, B, \text{1ingr}\Big(\text{Essnt}(B)\Big)\Big)\Big)$

E55 $[ABCDEFG]{::}A\ \varepsilon\ \text{rspcnstp}(B,C,D,E,F,G)$[60]$.\equiv{.:.}{}^1[\exists H].A\ \varepsilon\ \text{rspvarp}(B,C,D,E,$ $H).$

${}^2F\ \varepsilon\ \text{cnf}(A).$

${}^3F\ \varepsilon\ \text{homosemp}(A,C).$

${}^4G\ \varepsilon\ \text{expr}.$

${}^5\Big(\text{vrb} \cap \text{in}(G)\Big) \infty \Big(\text{vrb} \cap \text{in}(D$

$)\Big){.:}$

${}^6[HI]{:}H\ \varepsilon\ \text{vrb}.H\ \varepsilon\ \text{in}(D).I\ \varepsilon$ $\text{vrb}.I\ \varepsilon\ \text{in}(G).\Big(\text{vrb} \cap \text{in}(D) \cap \text{prcd}(H)\Big) \infty \Big(\text{vrb} \cap \text{in}(G) \cap \text{prcd}(I)\Big).H\ \varepsilon \sim\Big(\text{cnf}(I)\Big).$ $\supset.[\exists J].H\ \varepsilon\ \text{rspvarp}(B,C,D,E,J)$

E55.0 $\text{C6}\ \varepsilon\ \text{rspcnstp}\Big(\text{4ingr(C26),C6,C6,Eqvl1(C26),C6,Eqvl1(C1)}\Big)$

E55.1 $\text{C6}\ \varepsilon \sim\Big(\text{rspcnstp(Samp,C6,C6,Samp,C6,C6)}\Big)$

E55.2 $\text{C6}\ \varepsilon \sim\Big(\text{rspcnstp}\Big(\text{4ingr(C26),C6,C6,Eqvl1(C26),Eqvl1(C1),Eqvl1(C1)}\Big)\Big)$

E55.3 $\text{C6}\ \varepsilon \sim\Big(\text{rspcnstp}\Big(\text{4ingr(C26),C6,C6,Eqvl1(C26),Eqvl1(C7),Eqvl1(C1)}\Big)\Big)$

E55.4 $[A]{:}A\ \varepsilon$ "$\llcorner p \lrcorner \ulcorner \phi(pp)\urcorner$"$.\supset.\text{Eqvl1(C1)}\ \varepsilon \sim\Big(\text{rspcnstp}\Big(\text{2ingr}(A)\text{,C1,C1,}\ A\text{,Eqvl1}\Big($

$\text{C1),Ccl}\Big(\text{3ingr(C2)} \cup \text{4ingr(C2)} \cup \text{5ingr(C2)} \cup \text{6ingr(C2)} \cup \text{Uingr(C2)}\Big)\Big)\Big)$

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[59] This example assumes that in constructing system *C* we have modified thesis *C95* and all subsequent theses, replacing any words equiform with 'ϕ' by means of words equiform with 'Λ'; thesis *A* in the resulting system corresponds to thesis *C96* in system *C*.

[60] *A* is a term equiform with, and in the same semantic category as *F*, corresponding to a variable bound by *B*, relative to *C*, in *D* and *E* respectively, where *G* differs from *D* only at places corresponding to words bound by *B* in *E*.

E55.5 $\mathrm{C6}\ \varepsilon \sim\Big(\mathrm{rspcnstp}\Big(\mathrm{4ingr}\,(\mathrm{C26}), \mathrm{C6}, \mathrm{C6}, \mathrm{Eqvl1}\,(\mathrm{C26}), \mathrm{C6}, \mathrm{Cmpl}\Big(\mathrm{Eqvl1}\,(\mathrm{C1}) \cup \mathrm{Eqvl2}(\mathrm{C1})\Big)\Big)\Big)$

E55.6 $\mathrm{5ingr}(\mathrm{C24})\ \varepsilon \sim\Big(\mathrm{rspcnstp}\Big(\mathrm{2ingr}(\mathrm{C25}),\mathrm{C24},\mathrm{C24},\mathrm{C25}, \mathrm{5ingr}(\mathrm{C24}),\mathrm{C23}\Big)\Big)$

E56 $[ABC]::A\ \varepsilon\ \mathrm{cnsqverfp}(B,C)$[61] $.\equiv.\therefore{}^1C\ \varepsilon\ \mathrm{thp}(B):$

$^2[\exists DEF].D\ \varepsilon\ \mathrm{rspvarp}(E,B,C,A,F)\therefore$

$^3[CDEF]:C\ \varepsilon\ \mathrm{thp}(B).D\ \varepsilon\ \mathrm{rspvarp}(E,B,C,A,F).\supset.[\exists GH].D\ \varepsilon\ \mathrm{plenp}(B,G,H)\therefore$

$^4[CDEFGHI]:C\ \varepsilon\ \mathrm{thp}.D\ \varepsilon\ \mathrm{rspvarp}(E,B,C,A,F).G\ \varepsilon\ \mathrm{plenp}(B,H,I).G\ \varepsilon\ \mathrm{homosemp}(D,B).\supset.[\exists JK].J\ \varepsilon\ \mathrm{thp}(B).K\ \varepsilon\ \mathrm{rspcnstp}(E,B,J,A,G,C)$

E56.0 $\mathrm{C25}\ \varepsilon\ \mathrm{cnsqverfp}(\mathrm{C24},\mathrm{C24})$

E56.1 $\mathrm{C25}\ \varepsilon \sim\big(\mathrm{cnsqverfp}(\mathrm{C1},\mathrm{C24})\big)$

E56.2 $\mathrm{C25}\ \varepsilon \sim\big(\mathrm{cnsqverfp}(\mathrm{C1},\mathrm{C1})\big)$

E56.3 $\mathrm{C25}\ \varepsilon \sim\big(\mathrm{cnsqverfp}(\mathrm{C4},\mathrm{C4})\big)$

E56.4 $\mathrm{C25}\ \varepsilon \sim\big(\mathrm{cnsqverfp}(\mathrm{C23},\mathrm{C23})\big)$

E57 $[ABC]::A\ \varepsilon\ \mathrm{cnsqrejp}(B,C)$[62] $.\equiv.\therefore{}^1C\ \varepsilon\ \mathrm{thp}(B):$

$^2[\exists DEF].D\ \varepsilon\ \mathrm{rspvarp}\big(E,B,\mathrm{Negt}(C),\mathrm{Negt}(A),F\big)\therefore$

$^3[D]:D\ \varepsilon\ \mathrm{cnvar}\big(D,\mathrm{Negt}(A)\big).\supset.[\exists EFGHI].E\ \varepsilon\ \mathrm{extvarp}\big(D,\mathrm{Negt}(A),B,F,G,H,I\big)$

E57.0 $\mathrm{C26}\ \varepsilon\ \mathrm{cnsqrejp}(\mathrm{C25},\mathrm{C1})$

E57.1 $\mathrm{C26}\ \varepsilon \sim\Big(\mathrm{cnsqrejp}\big(\mathrm{C25},\mathrm{Eqvl1}(\mathrm{C2})\big)\Big)$

E57.2 $\mathrm{C26}\ \varepsilon \sim\big(\mathrm{cnsqrejp}(\mathrm{C25},\mathrm{C3})\big)$

E57.3 $\mathrm{C26}\ \varepsilon \sim\big(\mathrm{cnsqrejp}(\mathrm{C24},\mathrm{C1})\big)$

E58 $[ABCab]\because A\ \varepsilon\ \mathrm{resp}(B,C,a,b)$[63] $.\equiv::{}^1A\ \varepsilon\ \mathrm{Cmpl}\big(\mathrm{1ingr}(A) \cup a\big).$

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[61] A is a consequence by verification, relative to B, by means of C.

[62] A is a consequence by rejection, relative to B, by means of C.

[63] A is a substitution, corresponding to B, of part of C, by means of a and of b.

^{2}1ingr$(A)\ \varepsilon$ trm.

$^3B\ \varepsilon$ psubst$\Big($Eqvl1$\big($Essnt$(C)\big), C, b\Big)$::

$^4[D]$.˙.$D\ \varepsilon\ a.\lor.D\ \varepsilon$ vrb $\cap$ in$\Big($Eqvl2$\big($Essnt$(C)\big)\Big)$

$\cap$ scd$\Big($1ingr$\Big($Eqvl2$\big($Essnt$(C)\big)\Big)\Big)$:$\supset$.Cmpl$(a)\ \varepsilon$ psubst$\Bigg($Cmpl$\Bigg($vrb $\cap$ in$\Big($Eqvl2$\Big($Essnt$($

$C)\Big)\Big)$ $\cap$ scd$\Big($1ingr$\Big($Eqvl2$\big($Essnt$(C)\big)\Big)\Big)\Bigg), C, a\Bigg)$::

$^5[DEFG]$:$D\ \varepsilon\ a.E\ \varepsilon\ b.F\ \varepsilon$ cnvar$(G, C).F\ \varepsilon$ in

$\Big($Eqvl1$\big($Essnt$(C)\big)\Big).G\ \varepsilon$ in$\Big($Eqvl2$\big($Essnt$(C)\big)\Big).\big(b \cap$ prcd$(E)\big)$ ∞ $\Bigg($vrb $\cap$ in$\Big($Eqvl1$\Big($

Essnt$(D)\Big)\Big)$ $\cap$ prcd$(F)\Bigg).\big(a \cap$ prcd$(D)\big)$ ∞ $\Bigg($vrb $\cap$ in$\Big($Eqvl2$\big($Essnt$(C)\big)\Big)$ $\cap$ prcd(G) $\cap$

scd$\Big($1ingr$\Big($Eqvl2$\big($Essnt$(C)\big)\Big)\Big)\Bigg)$.$\supset$.$D\ \varepsilon$ cnf(E)

E58.0 Eqvl1(C1) ε resp$\big($C1,C5,square circles[64],vrb $\cap$ in(C1)$\big)$

E58.1 1ingr(C98) ε $\sim\Big($resp$\Big($C90,C95,vrb $\cap$ in(C98) $\cap$ scd$\big($1ingr(C98)$\big)$,vrb $\cap$ in(

C90)$\Big)\Big)$

E58.2 2ingr(C1) ε $\sim\Big($resp$\big($C1,C5,square circles,vrb $\cap$ in(C1)$\big)\Big)$

E58.3 C6 ε $\sim\Big($resp$\big($C7,C5,square circles,vrb $\cap$ in(C1)$\big)\Big)$

E58.4 C6 ε $\sim\Big($resp$\big($C1,C5,C6,vrb $\cap$ in(C1)$\big)\Big)$

E58.5 C44 ε $\sim\Big($resp$\Big($C7,C43,vrb $\cap$ in(C44) $\cap$ scd$\big($1ingr(C44)$\big)$,vrb $\cap$ in(C7)$\Big)\Big)$

E59 $[ABC]$:$A\ \varepsilon$ resp(B, C).$\equiv$.$^1[\exists ab]A\ \varepsilon$ resp(B, C, a, b)

E59.0 Eqvl1(C1) ε resp(C1,C5)

E59.1 C1 ε $\sim\big($resp(C1,C5)$\big)$

E60 $[ABCDEFG]$.˙.$A\ \varepsilon$ divsigp(B, C, D, E, F, G)[65].$\equiv$:$^1A\ \varepsilon$ resp(E, D).

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[64] The expression ‘square circles’ is equivalent to any other name which names no objects.

[65] A is an example beginning with a constant equiform with B and belonging to the same semantic category relative to C, decided in the opposite way from E, which corresponds to it according to D, by means of F and G.

$^2B\ \varepsilon$ cnf$\Big($1ingr$(A)\Big)$.

$^3B\ \varepsilon$ homosemp$\Big($1ingr$(A), C\Big)$.

$^4F\ \varepsilon$ thp(C).

$^5G\ \varepsilon$ thp(C):

$^6A\ \varepsilon$ Id$(F).E\ \varepsilon$ Negt$(G).\vee.A\ \varepsilon$ Negt$(F).E\ \varepsilon$ Id(G)

E60.0 Eqvl1(C1) ε divsigp$\Big($Eqvl1(C1),C1,C5,C1,C1,C1$\Big)$

E60.1 Eqvl1(C1) $\varepsilon \sim\Big($divsigp$\Big($Eqvl1(C1),C1,C2,C1,C1,C1$\Big)\Big)$

E60.2 Eqvl1(C1) $\varepsilon \sim\Big($divsigp(C6,C6,C5,C1,C1,C1)$\Big)$

E60.3 $[A]{:}A\ \varepsilon$ thp[66]$.A\ \varepsilon$ "Λ(V)"$.\supset.$Eqvl1(C1) $\varepsilon \sim\Big($divsigp$\Big($1ingr$(A), A$,C5,C1,C1, C1$)\Big)$

E60.4 5ingr(C2) $\varepsilon \sim\Big($divsigp$\Big($Eqvl1(C1),C2,C5,C1,Eqvl1(C2),C1$\Big)\Big)$

E60.5 Eqvl1(C1) $\varepsilon \sim\Big($divsigp$\Big($Eqvl1(C1),C1,C5,Eqvl1(C2),C1,Eqvl1(C2)$\Big)\Big)$

E60.6 C6 $\varepsilon \sim\Big($divsigp(C6,C6,C5,C1,C6,C1)$\Big)$

E61 $[AB]{::}A\ \varepsilon$ legdefp(B)[67]$.\equiv{::}^1[CD]{:}C\ \varepsilon$ qntf$.C\ \varepsilon$ in$(A).D\ \varepsilon$ in$(C).\supset.D\ \varepsilon \sim\Bigg(cnf\Bigg($ 1ingr$\Big($Eqvl2$\Big($Essnt$(A)\Big)\Big)\Bigg)\Bigg)$.∴

$^2[C]$.∴$C\ \varepsilon$ trm$.C\ \varepsilon$ in$\Big($Eqvl1$\Big($Essnt$(A)\Big)\Big).\supset{:}[\exists D].D\ \varepsilon$ qntf$.D\ \varepsilon$ in$(A).C\ \varepsilon$ int$(D).\vee.[\exists DE].D\ \varepsilon$ in$(A).C\ \varepsilon$ var$(E, D).\vee.C\ \varepsilon$ constp$(B, A){::}$

$^3[CD]{:}D\ \varepsilon$ qntf$.D\ \varepsilon$ in$(A).C\ \varepsilon$ int$(D).\supset.[\exists EF].E\ \varepsilon$ in$(A).F\ \varepsilon$ var(C, E).∴

$^4[CDE]{:}C\ \varepsilon$ int$\Big($Qntf$(A)\Big).E\ \varepsilon$ prntm$\Big($Essnt$(A)\Big).D\ \varepsilon$ arg $(E).\supset.[\exists F].F\ \varepsilon$ in$(D).F\ \varepsilon$ var$(C, A){::}$

$^5[CDE]$.∴$C\ \varepsilon$ in$\Big($Eqvl1$\Big($Essnt$(A)\Big)\Big).E\ \varepsilon$ in$(A).D\ \varepsilon$ cnvar $(C, E).D\ \varepsilon$ in$\Big($Eqvl1$\Big($Essnt$(A)\Big)\Big).\supset{:}D\ \varepsilon$ Id$(C).\vee.[\exists FG].D\ \varepsilon$ quasihomosemp$(C, B, A, F, G){::}$

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[66] This example assumes that in constructing system *C* we have modified thesis *C29* and any subsequent theses containing words equiform with '⫞', replacing all words equiform with '⫞' by means of words equiform with 'Λ'; thesis *A* in the resulting system corresponds to thesis *C31* in system *C*.

[67] *A* is acceptable as a definition immediately after thesis *B*.

$^{6}[C]{:}C\ \varepsilon\ \mathrm{gnrl}.C\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon \sim\!\big(\mathrm{Id}(A)\big).\supset.[\exists DEFG].D\ \varepsilon\ \mathrm{homosemp}(B,B).E\ \varepsilon\ \mathrm{thp}(B).F\ \varepsilon\ \mathrm{in}(E).G\ \varepsilon\ \mathrm{in}(A).D\ \varepsilon\ \mathrm{Anarg}(C,F,G){::}$

$^{7}[CD]{\therefore}C\ \varepsilon\ \mathrm{gnrl}.C\ \varepsilon\ \mathrm{in}(A).D\ \varepsilon\ \mathrm{Essnt}(C).\supset{:}D\ \varepsilon\ \mathrm{vrb}.\vee.[\exists E].E\ \varepsilon\ \mathrm{frp}(B).D\ \varepsilon\ \mathrm{genfnct}(E){::}$

$^{8}[C]{\therefore}C\ \varepsilon\ \mathrm{fnct}.C\ \varepsilon\ \mathrm{in}\Big(\mathrm{Eqvl1}\big(\mathrm{Essnt}(A)\big)\Big).\supset{:}[\exists D].D\ \varepsilon\ \mathrm{gnrl}.D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{Essnt}(D).\vee.[\exists DE].C\ \varepsilon\ \mathrm{fnctp}(B,A,D,E){::}$

$^{9}[C]{:}C\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).\supset.[\exists D].D\ \varepsilon\ \mathrm{arg}(C)\ \therefore$

$^{10}[CD]{:}C\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).D\ \varepsilon\ \mathrm{arg}(C).\supset.[\exists E].D\ \varepsilon\ \mathrm{var}(E,A){\therefore}$

$^{11}[CD]{:}C\ \varepsilon\ \mathrm{trm}.C\ \varepsilon\ \mathrm{in}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).D\ \varepsilon\ \mathrm{trm}.D\ \varepsilon\ \mathrm{in}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).C\ \varepsilon\ \mathrm{cnf}(D).\supset.C\ \varepsilon\ \mathrm{Id}(D){\therefore}$

$^{12}[CD]{:}C\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).D\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).C\ \varepsilon\ \mathrm{simprntm}(D).\supset.C\ \varepsilon\ \mathrm{Id}(D){\therefore}$

$^{13}[CDE]{:}C\ \varepsilon\ \mathrm{1prntmp}(B,A,D,E).\mathrm{Uingr}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big)\ \varepsilon\ \mathrm{in}(C).\supset.C\ \varepsilon\ \mathrm{simprntm}(E){\therefore}$

$^{14}[CDEFG]{:}C\ \varepsilon\ \mathrm{2prntmp}(B,A,D,E,F,G).G\ \varepsilon\ \mathrm{in}(A).\mathrm{Uprcd}(G)\ \varepsilon\ \mathrm{in}(C).\supset.C\ \varepsilon\ \mathrm{simprntm}(E){\therefore}$

$^{15}[CDE]{:}C\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).\mathrm{Uingr}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big)\ \varepsilon\ \mathrm{in}(C).D\ \varepsilon\ \mathrm{thp}(B).E\ \varepsilon\ \mathrm{in}(D).C\ \varepsilon\ \mathrm{simprntm}(E).\supset.[\exists FG].C\ \varepsilon\ \mathrm{1prntmp}(B,A,F,G){\therefore}$

$^{16}[CDEF]{:}C\ \varepsilon\ \mathrm{prntm}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big).D\ \varepsilon\ \mathrm{prntm}.D\ \varepsilon\ \mathrm{in}(A).\mathrm{Uprcd}(D)\ \varepsilon\ \mathrm{in}(C).E\ \varepsilon\ \mathrm{thp}(B).F\ \varepsilon\ \mathrm{in}(E).C\ \varepsilon\ \mathrm{simprntm}(F).\supset.[\exists GHI].C\ \varepsilon\ \mathrm{2prntmp}(B,A,G,H,I,D){::}$

$^{17}[CDEF]{\therefore}C\ \varepsilon\ \mathrm{qntf}.C\ \varepsilon\ \mathrm{in}(A).D\ \varepsilon\ \mathrm{int}(C).E\ \varepsilon\ \mathrm{thp}(B).F\ \varepsilon\ \mathrm{in}(E).F\ \varepsilon\ \mathrm{cnf}(D).\supset{:}[\exists G].G\ \varepsilon\ \mathrm{qntf}.G\ \varepsilon\ \mathrm{in}(E).F\ \varepsilon\ \mathrm{int}(G).\vee.[\exists GH].G\ \varepsilon\ \mathrm{in}(E).F\ \varepsilon\ \mathrm{var}(H,G){::}$

$^{18}[CDE]{:}C\ \varepsilon\ \mathrm{thp}(B).D\ \varepsilon\ \mathrm{qntf}.D\ \varepsilon\ \mathrm{in}(C).E\ \varepsilon\ \mathrm{int}(D).\supset.E\ \varepsilon \sim\!\Bigg(\mathrm{cnf}\bigg(\mathrm{1ingr}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big)\bigg)\Bigg){\therefore}$

$^{19}[CD]{:}D\ \varepsilon\ \mathrm{in}(A).C\ \varepsilon\ \mathrm{in}\Big(\mathrm{Eqvl1}\big(\mathrm{Essnt}(A)\big)\Big).C\ \varepsilon\ \mathrm{cnvar}(C,D).\supset.[\exists EFGHI].E\ \varepsilon\ \mathrm{extvarp}(C,A,B,F,G,H,I){\therefore}$

$^{20}[CDEF]{:}C\ \varepsilon\ \mathrm{thp}(B).D\ \varepsilon\ \mathrm{in}(C).F\ \varepsilon\ \mathrm{in}(A).E\ \varepsilon\ \mathrm{An}\Big(\mathrm{1ingr}\Big(\mathrm{Eqvl2}\big(\mathrm{Essnt}(A)\big)\Big), D, F\Big).E\ \varepsilon\ \mathrm{constp}(B,C).\supset.[\exists GHIJ].G\ \varepsilon\ \mathrm{divsigp}(E,B,A,H,I,J)$

E61.0 C5 ε legdefp(C4)

E61.1 $[A]{:}A\ \varepsilon$ “$\llcorner ps \lrcorner \ulcorner \varphi\big(\varphi(ps)s(p)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.2 $[A]{:}A\ \varepsilon$ “$\llcorner p \lrcorner \ulcorner \varphi\big(\varphi(pp){\dashv}(p)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C30)}\big)$

E61.3 $[A]{:}A\ \varepsilon$ “$\llcorner p \lrcorner \ulcorner \varphi\big(\varphi\text{-}(pp){\vdash\dashv}(p)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.4 $[A]{:}A\ \varepsilon$ “$\llcorner p \lrcorner \ulcorner \varphi\big(\varphi(p\llcorner qr \lrcorner \ulcorner q \urcorner){\vdash}(p)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.5 $[A]{:}A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \varphi\big(p\text{-}\varphi(pq)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.6 $[A]{:}A\ \varepsilon$ “$\llcorner pqr \lrcorner \ulcorner \varphi\Big(\varphi\big(\llcorner s \lrcorner \ulcorner \varphi\big(s(p)s\big)\urcorner \varphi(qr)\big)\phi(pqr)\Big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C79)}\big)$

E61.7 $[AB]{:}A\ \varepsilon\ \mathrm{thp}$[68]$.A\ \varepsilon$ “$\llcorner fp \lrcorner \ulcorner \varphi\big(f(p)\phi\underset{1}{<}fp\underset{1}{>}\big)\urcorner$”$.B\ \varepsilon$ “$\llcorner pqr \lrcorner \ulcorner \varphi\Big(\varphi\big(\phi\underset{1}{<}\llcorner u \lrcorner \ulcorner u \urcorner p\underset{1}{>}\varphi(qr)\big)\psi(pqr)\Big)\urcorner$”$.\supset.B\ \varepsilon \sim\big(\mathrm{legdefp}(A)\big)$

E61.8 $[AB]{:}A\ \varepsilon\ \mathrm{thp}$[69]$.A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \varphi\big(\varphi(pq)\varphi\underset{1}{(}p\underset{1}{)}(q)\big)\urcorner$”$.B\ \varepsilon$ “$\llcorner pqr \lrcorner \ulcorner \varphi\Big(\varphi\Big(\varphi\big(\llcorner s \lrcorner \ulcorner \varphi\underset{1}{(}s\underset{1}{)}\urcorner p\big)\varphi(qr)\Big)\phi(par)\Big)$”$.\supset.B\ \varepsilon \sim\big(\mathrm{legdefp}(A)\big)$

E61.9 $[A]{:}A\ \varepsilon$ “$\llcorner pqr \lrcorner \ulcorner \varphi\big(\varphi(pq)(qr)\phi(pqr)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.10 $[A]{:}A\ \varepsilon$ “$\llcorner p \lrcorner \ulcorner \varphi\big(p{\dashv}(\llcorner)(p)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.11 $[A]{:}A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \varphi\big(\varphi(pq)\phi(pqr)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.12 $[A]{:}A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \varphi\big(\varphi(pq)\phi(ppq)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.13 $[A]{:}A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \varphi\big(\varphi(pq)\varphi(p)(q)\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

E61.14 $[A]{:}A\ \varepsilon$ “$\llcorner pq \lrcorner \ulcorner \varphi\big(\varphi(pq)\phi{<}pq{>}\big)\urcorner$”$.\supset.A\ \varepsilon \sim\big(\mathrm{legdefp(C25)}\big)$

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[68] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

[69] This example assumes that thesis A has been added to system C immediately after thesis *C99*.

E61.15 $[ABC]\colon A \;\varepsilon\; \mathrm{thp}(B)$[70]$.A \;\varepsilon\;$ “$\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\underset{1}{\phi}\underset{1}{(p)}(q)\big)\urcorner$”$.B \;\varepsilon\; \mathrm{thp}.B \;\varepsilon\;$ “$\phi\big(\underset{1}{\phi}\underset{1}{(\Lambda)}(\mathrm{V})\Lambda\big)$”$.C \;\varepsilon\;$ “$\llcorner pq \lrcorner \ulcorner \phi\big(\phi\text{-}(pq)\underset{2}{\phi\text{-}}\underset{2}{(p)}(q)\big)\urcorner$”$.\supset.C \;\varepsilon \sim\big(\mathrm{legdefp}(B)\big)$

E61.16 $[ABC]\colon A \;\varepsilon\; \mathrm{thp}(B)$[71]$.A \;\varepsilon\;$ “$\mathrm{V}(\Lambda)$”$.B \;\varepsilon\; \mathrm{thp}.B \;\varepsilon\;$ “$\llcorner f \lrcorner \ulcorner \phi\big(f(\Lambda)f(\Lambda)\big)\urcorner$”$.C \;\varepsilon\;$ “$\llcorner fp \lrcorner \ulcorner \phi\big(f(p)\phi(fp)\big)\urcorner$”$.\supset.C \;\varepsilon \sim\big(\mathrm{legdefp}(B)\big)$

E61.17 $A \;\varepsilon\;$ “$\llcorner pqr \lrcorner \ulcorner \phi\Big(\phi\big(\phi(pq)r\big)\phi(p)(qr)\Big)\urcorner$”$.\supset.A \;\varepsilon \sim\big(\mathrm{legdefp}(\mathrm{C29})\big)$

E61.18 $[A]\colon A \;\varepsilon\;$ “$\llcorner \Lambda \lrcorner \ulcorner \phi\Lambda\dashv(\Lambda)\big)\urcorner$”$.\supset.A \;\varepsilon \sim\big(\mathrm{legdefp}(\mathrm{C25})\big)$

E61.19 $[A]\colon A \;\varepsilon\;$ “$\llcorner q \lrcorner \ulcorner \phi\big(qp(q)\big)\urcorner$”$.\supset.A \;\varepsilon \sim\big(\mathrm{legdefp}(\mathrm{C25})\big)$

E61.20 $\mathrm{C29} \;\varepsilon \sim\big(\mathrm{legdefp}(\mathrm{C24})\big)$

E61.21 $[A]\colon A \;\varepsilon\;$ “$\llcorner pq \lrcorner \ulcorner \phi\big(\phi(pq)\underset{1}{\phi}(pq)\big)\urcorner$”$.\supset.A \;\varepsilon \sim\big(\mathrm{legdefp}(\mathrm{C25})\big)$

E62 $[ABCD]\therefore A \;\varepsilon\; \mathrm{1cnsqdefp}(B,C,D)$[72]$.\equiv\colon{}^{1}[\exists E].C \;\varepsilon\; \mathrm{primp}(E,B)\colon$

${}^{2}D \;\varepsilon\; \mathrm{thp}(B).$

${}^{3}A \;\varepsilon\; \mathrm{resp}(D,C).$

${}^{4}\mathrm{1ingr}(A) \;\varepsilon\; \mathrm{cnf}\Bigg(\mathrm{1ingr}\bigg(\mathrm{Eqvl2}\Big(\mathrm{Essnt}(C)\Big)\bigg)\Bigg)$

E62.0 $\mathrm{C6} \;\varepsilon\; \mathrm{1cnsqdefp}(\mathrm{C5,C5,C1})$

E62.1 $\mathrm{C6} \;\varepsilon \sim\big(\mathrm{1cnsqdefp}(\mathrm{C7,C7,C6})\big)$

E62.2 $\mathrm{C6} \;\varepsilon \sim\Big(\mathrm{1cnsqdefp}\big(\mathrm{C5,C5,Eqvl1(C2)}\big)\Big)$

E62.3 $\mathrm{C6} \;\varepsilon \sim\big(\mathrm{1cnsqdefp}(\mathrm{C5,C5,C2})\big)$

E62.4 $\mathrm{Eqvl1(C1)} \;\varepsilon \sim\big(\mathrm{1cnsqdefp}(\mathrm{C5,C5,C1})\big)$

E63 $[ABCD]\therefore A \;\varepsilon\; \mathrm{2cnsqdefp}(B,C,D).\equiv\colon{}^{1}[\exists E].C \;\varepsilon\; \mathrm{primp}(E,B)\colon$

${}^{2}D \;\varepsilon\; \mathrm{thp}(B).$

${}^{3}\mathrm{Negt}(A) \;\varepsilon\; \mathrm{resp}\big(\mathrm{Negt}(D),C\big).$

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[70] This example assumes that thesis A has been added to system C immediately after thesis *C99*, and that thesis B has been added immediately after thesis A.

[71] This example assumes that we have modified system C so that in thesis *C43* and in all subsequent theses, we have replaced any words equiform with ‘$\vdash\!\dashv$’ with words equiform with ‘V’; in the resulting system A corresponds to thesis *C44*, and B corresponds to thesis *C79*.

[72] A is an expression of the first kind which can be inferred, relative to B, from C and D.

^{4}1ingr$\Big($Negt$(A)\Big)$ ε cnf$\Bigg($1ingr$\Bigg($Eqvl2$\Big($Essnt$(C)\Big)$

$\Bigg)\Bigg)$

E63.0 C30 ε 2cnsqdefp(C29,C29,C1)

E63.1 C15 ε $\sim\Big($2cnsqdefp(C22,C22,C9)$\Big)$

E63.2 C30 ε $\sim\Bigg($2cnsqdefp$\Big($C29,C29,Eqvl1(C2)$\Big)\Bigg)$

E63.3 C30 ε $\sim\Big($2cnsqdefp(C29,C29,C6)$\Big)$

E63.4 C68 ε $\sim\Big($2cnsqdefp(C29,C29,C1)$\Big)$

E64 $[ABC]$:A ε 1compn(B,C)[73].$\equiv$.^{1}Eqvl1(A) ε cnf(B).
^{2}Eqvl2(A) ε cnf(C)

E64.0 C4 ε 1compn(C1,C2)

E64.1 C4 ε $\sim\Big($1compn(C2,C2)$\Big)$

E64.2 C4 ε $\sim\Big($1compn(C1,C1)$\Big)$

E65 $[ABC]$:A ε 2compn(B,C).$\equiv$.^{1}Eqvl1$\Big($Negt$(A)\Big)$ ε cnf(B).
^{2}Eqvl2$\Big($Negt$(A)\Big)$ ε cnf$\Big($Negt$(C)\Big)$

E65.0 C3 ε 2compn(C1,C1)

E65.1 C3 ε $\sim\Big($2compn(C2,C1)$\Big)$

E65.2 C3 ε $\sim\Big($2compn(C1,C2)$\Big)$

E66 $[ABC]$:A ε 3compn(B,C).$\equiv$.^{1}Eqvl1$\Big($Negt$(A)\Big)$ ε cnf$\Big($Negt$(B)\Big)$.
^{2}Eqvl2$\Big($Negt$(A)\Big)$ ε cnf(C)

E66.0 C9 ε 3compn(C1,C6)

E66.1 C9 ε $\sim\Big($3compn(C2,C6)$\Big)$

E66.2 C9 ε $\sim\Big($3compn(C1,C7)$\Big)$

E67 $[ABC]$:A ε 4compn(B,C).$\equiv$.^{1}Eqvl1(A) ε cnf$\Big($Negt$(B)\Big)$.

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[73] A is an expression of the first kind which can be obtained by combining B and C.

$$^2\mathrm{Eqvl2}(A)\ \varepsilon\ \mathrm{cnf}\Big(\mathrm{Negt}(C)\Big)$$

E67.0 C16 ε 4compn(C8,C9)

E67.1 C16 $\varepsilon \sim\Big($4compn(C9,C9)$\Big)$

E67.2 C20 $\varepsilon \sim\Big($4compn(C8,C9)$\Big)$

6.4. The directives of system C

The terms explained above allow us to express the directives of system C. Following the example of the three sets of directives published by Leśniewski, we state them as points of a single rule of procedure:

Under the presupposition that A is the last of the theses which already belongs to system C, one may add an expression B to the system immediately after A only if at least one of the nine following conditions is fulfilled:

(1) $[\exists CD].C\ \varepsilon\ \mathrm{thp}(A).D\ \varepsilon\ \mathrm{thp}(A).B\ \varepsilon\ \mathrm{1compn}(C,D)$ [Dir. *a*]

(2) $[\exists CD].C\ \varepsilon\ \mathrm{thp}(A).D\ \varepsilon\ \mathrm{thp}(A).B\ \varepsilon\ \mathrm{2compn}(C,D)$ [Dir. *b*]

(3) $[\exists CD].C\ \varepsilon\ \mathrm{thp}(A).D\ \varepsilon\ \mathrm{thp}(A).B\ \varepsilon\ \mathrm{3compn}(C,D)$ [Dir. *c*]

(4) $[\exists CD].C\ \varepsilon\ \mathrm{thp}(A).D\ \varepsilon\ \mathrm{thp}(A).B\ \varepsilon\ \mathrm{4compn}(C,D)$ [Dir. *d*]

(5) $B\ \varepsilon\ \mathrm{legdefp}(A)$ [Dir. *e*]

(6) $[\exists CD].B\ \varepsilon\ \mathrm{1cnsqdefp}(A,C,D)$ [Dir. *f*]

(7) $[\exists CD].B\ \varepsilon\ \mathrm{2cnsqdefp}(A,C,D)$ [Dir. *g*]

(8) $[\exists C].B\ \varepsilon\ \mathrm{cnsqverfp}(A,C)$ [Dir. *h*]

(9) $[\exists C].B\ \varepsilon\ \mathrm{cnsqrejp}(A,C)$ [Dir. *i*]

We can, of course, verify that system C has been constructed in a manner which satisfies these directives. But we should be careful to remember, while constructing proofs in the metatheory of system C, that there are ways of developing the system, based on the same axiom and conforming to the same directives, which diverge considerably from the system derived in chapter 4.

6.5. Alternative directives

In investigating system C I spent some time considering not only alternative formulations of the directives, but also directives different from those described above. It is not likely that Leśniewski's directives for computative protothetic were expressed in precisely these terms. If any of his directives differed from mine in respect to the actual theses one could add with their help, I believe the directive for rejection is most likely to have differed, while the directive for definition may have differed in some respects.

In directive *E57*, for example, condition 3 prevents the added thesis from introducing variables into a semantic category which previously contains only constants and, possibly, functions. This could easily be replaced by a condition which required the semantic category of all variables in the new thesis to have been exhausted. In system C this would mean that thesis *C26* could be added immediately after thesis *C6*, whereas under the constraints I have imposed it cannot be added until after thesis *C25*.

Such variations are not particularly significant, in that the resulting systems resemble system C very closely. There are, however, some very interesting systems which depart the model of the systems described by Leśniewski, in that they do not require that we limit the number of constants in any semantic category. We can produce such a system by changing three of the directives of system C:

(1) The directive for definitions is replaced by a directive corresponding precisely to the definition directive in system $\mathfrak{S}_5$.

(2) The rejection directive is modified by simply omitting condition 3 from explanation *E57*. This allows new variables to be introduced without restriction, much as in system $\mathfrak{S}_5$.

(3) The verification directive becomes much more complex. First, the method for determining whether or not a group of constants exhausts a semantic category need not refer to all constants in that category. Second, we do not need to require that all possible substitutions be theses, only substitutions with some limited group of constants which exhaust their respective semantic categories. Finally, a number of 'safeguards' imposed indirectly by the restrictions on definitions and by the restrictions on variables in system C no longer obtain; this requires further complications in the explanations of terms corresponding to 'plenp' above.

7. The Metatheory of Protothetic

Among the most fundamental topics from the metatheory of protothetic are the consistency and the completeness of the theory. Ideally we should construct 'gapless' proofs of these, referring explicitly to conditions specified in terminological explanations and to other required presuppositions. The resulting proofs would exhibit aspects of the explanations and of their contents and interrelation which are not apparent when the explanations have been merely stated. However, such proofs would be far too long to include in the present work: the proof of the consistency of system C alone refers to nearly every condition in every terminological explanation in the last chapter.

Leśniewski published a few examples of metatheoretical 'proofs'; he says of a typical 'proof' of this kind that he wants 'to outline here, without any claim to precision' some 'circumstances' of which he became aware, and that this outline explains the method by which he assured himself that certain metatheorems are true[1]. The 'proof' which this describes demonstrates the consistency and completeness of system $\mathfrak{S}$[2].

Despite their length, proofs in the metatheory of protothetic can be quite straightforward, and they may lead to a better understanding of the system to which they apply. For this reason it is appropriate to present at least an outline of some of the main proofs concerning system C. In so doing I shall speak very informally, and with no claim to precision.

7.1. All theses are meaningful

Every thesis in a system of computative protothetic is meaningful; that is, $[AB]{:}A\ \varepsilon\ \mathrm{thp}(B).\supset.A\ \varepsilon\ \mathrm{propp}(B)$. The proof is complicated by the fact that some meaningful expressions cease to be meaningful as the system develops; that is, it is not the case for some systems of computative protothetic that $[ABC]{:}A\ \varepsilon\ \mathrm{propp}(B).B\ \varepsilon\ \mathrm{thp}(C).\supset.A\ \varepsilon\ \mathrm{propp}(C)$. In system C, for example, the expression '$\llcorner\mathrm{V}\lrcorner\ulcorner\mathrm{V}\urcorner$' is meaningful relative to thesis *C4*, but it is not meaningful relative to thesis *C5*; the second case violates condition 9 of explanation *E44a*. Informally we may say that, by defining a constant equiform with 'V', we prevent words of this shape from being used subsequently as legitimate variables.

The proof defines a property of being a thesis relative to which that thesis and all previous theses are meaningful:

$$[A]{::}A\ \varepsilon\ \mathrm{propthp}.\equiv{.\!:\!.}A\ \varepsilon\ \mathrm{thp}{.\!:\!.}[B]{:}B\ \varepsilon\ \mathrm{thp}(A).\supset.B\ \varepsilon\ \mathrm{propp}(A)$$

Axiom *C1* possesses this property, and it is inherited by all theses added to system C in accordance with the directives. Most of the terms in explanation *E44a* allow us to prove a lemma such as these:

$$[ABC]{:}A\ \varepsilon\ \mathrm{frp}(B).B\ \varepsilon\ \mathrm{thp}(C).\supset.A\ \varepsilon\ \mathrm{frp}(C)$$
$$[ABCD]{:}A\ \varepsilon\ \mathrm{constp}(B,C).B\ \varepsilon\ \mathrm{thp}(D).\supset.A\ \varepsilon\ \mathrm{constp}(D,C)$$
$$[ABCDEF]{:}A\ \varepsilon\ \mathrm{fnctp}(B,C,D,E).B\ \varepsilon\ \mathrm{thp}(F).\supset.A\ \varepsilon\ \mathrm{fnctp}(F,C,D,E)$$

---

[1] Leśniewski29, p. 16.

[2] Leśniewski29, pp. 15–30.

With the help of such lemmas, we see that all conditions of *E44a* which are true with respect to the earlier thesis are also true with respect to the later thesis, except condition 9. Moreover, in the case of condition 9, only directive e allows us to introduce new shapes of constants, while condition 19 of *E61* assures us that these new constants can never invalidate any variables in theses already belonging to the system. Hence we can prove that, if A is the last thesis belonging to the system, and B is a new thesis added to the system immediately after A in accordance with the directives, then

$$[C]{:}A\ \varepsilon\ \mathrm{propthp}.C\ \varepsilon\ \mathrm{thp}(A).\supset.C\ \varepsilon\ \mathrm{propp}(B)$$

We must also prove that, for each of the directives of system C, if A is the last thesis belonging to the system, and B is a new thesis added to the system immediately after A in accordance with the given directive, and A is propthp, then B is propp(B). It then follows that every thesis in system C is propthp, and therefore that every thesis in system C is meaningful relative to itself and to all theses of system C which follow it.

7.2. System C is consistent

A system of protothetic is consistent provided it does not contain two theses, one of which is equiform with the expression negated in the other. To prove this, we define the auxiliary property possessed by a thesis relative to which the system is consistent:

$$[A]{::}A\ \varepsilon\ \mathrm{cnsisth}.\equiv.{\because}A\ \varepsilon\ \mathrm{thp}.{\because}[BC]{:}B\ \varepsilon\ \mathrm{thp}(A).C\ \varepsilon\ \mathrm{thp}(A).\supset.B\ \varepsilon\ \sim\Big(\mathrm{cnf}\Big(\mathrm{Negt}(C)\Big)\Big)$$

Axiom *C1* of system C possesses this property, and it is hereditary with respect to the directives of system C.

We can think about this more clearly if we consider the first pair of inconsistent theses, comprising a thesis A which is not cnsisth, and which follows only theses which are cnsisth, and a thesis which is the first thesis preceding A which contradicts A; we call this thesis B if it is equiform with the expression A, or C if A is equiform with the expression which it negates.

Now, how can the first inconsistent thesis be added to system C? If any inconsistent thesis is added to the system, we can show that there must already have existed in the system a pair of inconsistent theses. In showing this, the following lemmas are useful:

Exhaustion Lemma. Once a variable is introduced into a semantic category, no further constants can be defined in that category. Although each of the nine directives may allow us to add a thesis containing a variable, only the verification directive h allows us to introduce a variable into a category not previously containing one. Note in particular condition 3 in *E57* and condition 20 in *E61*, which exist precisely to impose this condition on directives i and e respectively. If a new constant is defined, condition 21 of *E61* requires that it must differ from each previous constant in the same semantic category by having at least one corresponding critical expression decided in a different way. On the other hand, *E51* requires that a constant must exist for each possible combination of affirmed or negated critical theses. One could add a new definition legitimately if one defined a new constant in the semantic category of one of its arguments, but condition 20 in *E61* guarantees that every argument of a newly defined term must belong to a category into which variables have already been introduced; that is, if any definition violates the exhaustion lemma, then some previous definition must have violated the lemma. Hence no definition is the first to violate this lemma. Hence no definition violates this lemma.

Double Negation Lemma. It cannot be the case that the first pair of inconsistent theses contains a negation and a double negation; that is, that one of them, B, is of the type '$\phi(\alpha\Lambda)$', while the other thesis C is of the type '$\phi\big(\phi(\alpha\Lambda)\Lambda\big)$'. For C cannot be the axiom, which is not of this form. Hence it must be added by one of the directives. If it was added by one of the directives a or c, then there was a previous thesis of the type 'Λ', which contradicts axiom *C1*. If C was added by directive b, then there was a previous thesis of the type 'α', which contradicts B. If C was added by directive d, then there was a previous thesis equiform with C, which contradicts B. C was not added by directive e, since it violates condition 2 of *E61*. C was not added by either of the directives f or g, since in either case we should require a previous 'definition' of the type '$\llcorner pq \lrcorner \ulcorner \phi\big(\beta\phi(pq)\big)\urcorner$', and this violates condition 6 of *E46*. C was not added by directive h, since it is not a generalisation. Finally C was not added by directive i, since it is not the negation of a generalisation. Hence there is no such thesis as C in the first pair of inconsistent theses.

The first inconsistent thesis A cannot be added by directive a. If it were, then it would have the form '$\phi(\alpha\beta)$', where there are previous theses of types 'α' and 'β'. The previous thesis of type 'β' cannot be equiform with 'Λ', since this would contradict the axiom. Hence the given thesis does not have the form of a negation, and the other thesis C of the first inconsistent pair is of the type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$'. Thesis C does not have the form of axiom *C1* or of a thesis added by directives h or i. C was not added by either of the directives f or g, since in either case we should require a previous 'definition' of the type '$\llcorner pq \lrcorner \ulcorner \phi\big(\gamma\phi(pq)\big)\urcorner$', and this violates condition 6 of *E46*. C was not added by directive e, since it violates condition 2 of *E61*. If there is a thesis preceding A of one of the types '$\phi(\alpha\Lambda)$' or '$\phi(\alpha\Lambda)$', then A is not the first inconsistent thesis, so C was not added by any of the directives b, c, or d. Hence C was added by directive a, and there was a previous thesis of the type 'Λ', which contradicts the axiom. Hence A is not the first inconsistent thesis.

The first inconsistent thesis A cannot be added by directive b. If it were, it would be of the type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$', where there are previous theses of types 'α' and '$\phi(\beta\Lambda)$'; we shall call the last thesis D. By the double negation lemma, the other thesis B of the first inconsistent pair is of the type '$\phi(\alpha\beta)$'. Moreover, by the double negation lemma, the expression 'β' is not equiform with 'Λ'. Therefore thesis B does not have the form of the axiom or of a thesis added by one of the directives b, c, g, h, or i. B was not added by directive f, since this would require a previous 'definition' of the type '$\llcorner pq \lrcorner \ulcorner \phi\big(\gamma\phi(pq)\big)\urcorner$', which violates condition 6 of *E46*. B was not added by directive d unless there was an earlier thesis of the type '$\phi(\alpha\Lambda)$' inconsistent with the earlier thesis of the type 'α'. B was not added by directive a unless there was an earlier thesis of the type 'β' inconsistent with thesis D. Therefore B must have been added by directive e. We can then show that the first thesis equiform with D must have been added by directive f, and that there was a previous thesis of the type '$\phi(\alpha\Lambda)$'.

The first inconsistent thesis A cannot be added by directive c. If it were, it would be of the type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$', where there are previous theses of types '$\phi(\alpha\Lambda)$' and 'β'; we shall call this last thesis D. By the double negation lemma, the other thesis B of the first inconsistent pair is of the type '$\phi(\alpha\beta)$'. Moreover, by the double negation lemma, the expression 'β' is not equiform with 'Λ'. Therefore thesis B does not have the form of the axiom or of a thesis added by one of the directives b, c, g, h, or i. B was not added by directive f, since this would require a previous 'definition' of the type '$\llcorner pq \lrcorner \ulcorner \phi\big(\gamma\phi(pq)\big)\urcorner$', which violates condition 6 of *E46*. B was not added by directive d unless there was an earlier thesis of the type '$\phi(\beta\Lambda)$' inconsistent with D. B was not added by directive a unless there was an earlier thesis of the type 'α' inconsistent with the earlier thesis of the type '$\phi(\alpha\Lambda)$'. Hence B was added by directive e. From this we can show that thesis D must have been added by directive f, and hence that there is an earlier thesis of the type 'α'.

The first inconsistent thesis A cannot be added by directive d. For if it were, it would have the form '$\phi(\alpha\beta)$', where there are earlier theses of types '$\phi(\alpha\Lambda)$' and '$\phi(\beta\Lambda)$'. The expressions of type 'β' cannot be equiform with 'Λ' unless there is an earlier thesis equiform with A; hence the first thesis contradicting A, thesis C, has the form '$\phi\big(\phi(\alpha\beta)\Lambda\big)$'. C does not have the form of the axiom or of a thesis added by one of the directives h or i. As in the discussion of directive a above, C cannot have been added by one of the directives e, f, or g without violating *E61* or *E46*. Because there is not previous thesis of type 'Λ', C cannot have been added by directive a or by directive c. If C was added by directive b, then there was an earlier thesis equiform with A. Hence C was added by directive d, and there was an earlier thesis equiform with C, which is not possible.

The first inconsistent thesis cannot be added by directive e without violating condition 2 of explanation *E61*.

The first inconsistent thesis A cannot be added by directive f. For if it were, it would be an expression of the type 'α', which is either a defined term or a function beginning with a defined term, and not having the form of an equivalence. Note that 'α' is not equiform with 'Λ' unless the definition in question violates condition 2 of explanation *E61*. The corresponding substitution of the *definiens* of the definition is a previous thesis of the type 'β', as required by directive f. In either case the other thesis C of the first inconsistent pair must have the form '$\phi(\alpha\Lambda)$'. C is neither the axiom nor added by one of the directives a, b, e, f, h, or i. And C cannot have been added by directives c or d unless there was an earlier thesis equiform with C. Therefore C was added by directive g, and there was an earlier thesis negating the corresponding substitution of the *definiens*, that is, a thesis of the type '$\phi(\beta\Lambda)$'.

Similarly the first inconsistent thesis cannot be added by directive g, because the other thesis B of the first inconsistent pair must have been added by directive f, and there would be a previous contradiction.

The first inconsistent thesis A cannot be added by directive h. For if it were, it would have the form of a generalisation, and the other thesis C of the first inconsistent pair must be the negation of this generalisation. C cannot, as a matter of form, be the axiom or added by directives b, c, e, f, g, or h. If C was added by one of the directives a or d, then there is an earlier pair of contradictory theses. Therefore C was added by directive i, and there are two previous theses, one having the form of the negation of a substitution of A with basic constants (required by i), and the other having the form of the same substitution of A (required by h). Hence there is an earlier pair of inconsistent theses.

The first inconsistent thesis A must therefore be added by directive i, and the other thesis B of the first inconsistent pair must be a generalisation negated by A. Thesis B cannot, as a matter of form, be the axiom or added by any of the directives a, b, c, d, f, g, or i. Moreover, if B was added by directive h, then there must be a previous pair of inconsistent theses which are some substitution of B with basic constants and the negation of the same substitution. (We know that new constants cannot be introduced after either of these theses is added to the system because of the exhaustion lemma; hence it is not possible to introduce a *different* substitution of B.) Therefore the other thesis B of the first inconsistent pair must have been added by directive e, the directive for definitions. Thesis A was therefore proved from a thesis D of the type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$', in which the expression 'α' is the substitution of the *definiens* of B with basic constants, while 'β' is the corresponding substitution of the *definiendum.* Thesis D, as a matter of form, is not the axiom nor added by any of the directives e, f, g, h, or i. And, since there cannot have been a previous thesis of the type 'Λ', thesis D cannot have been added by directive a. Directive d could not add thesis D unless there was an earlier thesis equiform with D, whose origin must also be accounted for. If thesis D was added by directive b, then there are earlier theses of types 'α' and '$\phi(\beta\Lambda)$', where the earliest thesis of the latter type must have been added by directive g, so that there was also an earlier thesis of the type '$\phi(\alpha\Lambda)$', and hence an earlier inconsistent pair of theses. Similarly thesis D could not have been added by directive c unless there was an earlier pair of inconsistent theses of types '$\phi(\alpha\Lambda)$' and 'α'.

There is accordingly no way to introduce an inconsistent thesis into system C.

7.3. System C is complete

The proof of the completeness of system C, unlike the proofs outlined in the last two sections, makes essential use of the concept of provability. Loosely speaking, a system of protothetic is complete if every meaningful expression can be either proved or disproved. Complications arise because the notion of meaningful expressions changes as the system develops.

More precisely, then, the completeness of a system of protothetic requires that

(1) At any point in the development of the system, any expression meaningful relative to the last thesis must be able to be proved or disproved. In computative systems this can be done without extending the notion of meaningful expressions at that point.

(2) We must be able to define at least one constant in any semantic category which can be generated from categories which already exist in the system.

(3) We must be able to define constants which exhaust any semantic category which exists in the system.

In other words, requirements 2 and 3 allow the notion of meaningful expression to be extended, while 3 also includes an extension of the traditional notion of 'functional completeness'.

The proof of completeness is related to the following important metatheorem:

Extensionality Lemma. Given a constant A for which all possible critical expressions have been decided, and an expression B in the same semantic category as A, and having all of its 'critical expressions' decided 'in the same way' as the corresponding critical expression for A, then for every decided meaningful expression containing A, we can decide 'in the same way' a corresponding meaningful expression in which one or more occurrences of A are replaced by B[3].

The extensionality lemma and requirement 1 of the completeness theorem are best proved together. That is, the two theorems can be proved relative to a given stage of development of system C, and then they are shown to retain their validity each time the notion of meaningful expression is extended.

Relative to axiom *C1*, there are two constants, but we cannot prove critical expressions for the term 'ϕ'. Every meaningful expression at this stage must contain at least one 'Λ', and the axiom decides meaningful expressions containing one or two of these terms. If all expressions containing at most n of these terms are decidable, then all expressions containing at most $n+1$ are decidable, since each of the latter consists of an equivalence between two decidable expressions, and depending on whether the arguments are equiform with provable theses or with expressions negated in provable theses, we can apply one of the directives a, b, c, or d to decide the equivalence.

The extensionality lemma can be proved relative to axiom *C1* by observing that only directives a, b, c, and d are available for adding new theses to the system without extending the notion of meaningful expression. Given that an expression containing one or more 'Λ' is decided in system C by a thesis C, we can trace its proof back to axiom *C1* through a sequence of applications of these four directives. An expression of type 'α' satisfies the requirements of the extensionality lemma if its negation is proved by a thesis B. From this, we can construct a proof parallel to the proof of C, in which 'α' may replace any number of terms 'Λ' in the intermediate theses. A 'Λ' is introduced into a thesis required in the proof of C by being in the axiom, by being the second argument of an equivalence introduced by directives b or c, or by corresponding to a 'Λ' in an earlier thesis. Thesis B replaces the first 'Λ' with 'α' in the axiom. Directive d can be used with thesis B to replace the second 'Λ' in the axiom, in thesis B, or in any negation. In this way theses can be proved parallel to any or all theses in C, in which any or every 'Λ' is replaced with 'α'.

Hence system C is complete and extensional relative to axiom *C1*.

The notion of meaningful expression in system C can be extended in two ways: by defining a new constant, and by introducing variables into an existing semantic category.

If the system is complete and extensional relative to its last thesis, and if we add a thesis A which introduces variables into an existing semantic category, then we can show that the system is complete and extensional relative to the new thesis. For every variable in any thesis belongs to a semantic category which is exhausted by existing 'basic constants'. Moreover no category can be exhausted before the category of sentences is exhausted by defining 'V' (or a synonym for 'V') and proving thesis *C6* (or an equivalent thesis).

The system is extensional and complete for all meaningful expressions which contain no generalisations, since such expressions are meaningful relative to the last thesis preceding

[3] The definition of critical expressions in *E47* and of being decided in the same way in *E49* do not apply here because the expression B is not restricted to being either a term or a function with terms as arguments. This is avoided by defining new terms by means of explanations analogous to those just mentioned but broader in their application.

A. Extensionality for all meaningful expressions follows because a parallel proof can always be constructed step by step in which a generalisation is substituted for 'Λ' or 'V'.

Now, if all meaningful expressions containing n or fewer generalisations are decidable, then any meaningful expression B containing exactly $n+1$ generalisations is decidable. For B contains at least one generalisation C which is not contained in a larger generalisation. To C there correspond a finite number of possible substitutions of C with basic constants, and we know that every variable in C belongs to an exhausted semantic category. Moreover every such substitution contains one generalisation less than C does, and hence n or fewer. Therefore every such substitution of C is decidable. If all can be added as theses, then we can prove a thesis equiform with C by directive h, the expression corresponding to B in which C is replaced by 'V' is decidable, and B is decidable by the extensionality lemma. On the other hand, if at least one substitution of C can be negated, then we can prove the negation of C by directive i, the expression corresponding to B in which C is replaced by 'Λ' is decidable, and B is decidable by the extensionality lemma.

It follows that any expression meaningful with respect to A is decidable, and hence that the system as extended by A is complete and extensional relative to A.

If the system is complete and extensional relative to its last thesis, and we add a definition A as the next thesis in accordance with directive e, then we can show that the system extended by definition A is complete and extensional relative to A.

First we observe that all critical expressions beginning with the newly defined term B are decidable. For every such critical expression is a legitimate substitution of the *definiendum* of A, with basic constants replacing any variables. The corresponding substitution of the *definiens* is meaningful relative to the last thesis preceding A, and is hence decidable. Therefore the critical expression can be decided by one of the directives f or g.

Once all such critical expressions have been decided, we can prove by directives a and d all possible substitutions of A with basic constants. If B is propositional, there is one equivalence which is equiform with A; otherwise we can prove a generalisation equiform with A by directive h.

If the extensionality lemma applies to any basic constants in the categories of the arguments of a constant C, then it applies to C. For the step by step construction of a parallel proof can be carried out except in the following cases:

(1) C is the term for equivalence, and it is introduced into a thesis by one of the directives a, b, c, d, g, or i. Since extensionality applies to propositional constants, the arguments of the equivalence can be decided and can replace the constants 'V' or 'Λ' in a decidable expression.

(2) C is the term for equivalence, and it is the main functor of a thesis introduced by directive e. We prove a thesis equiform with e as outlined above, and we construct the corresponding thesis step by step following this proof.

(3) C is a defined term introduced into its own definition. We prove a thesis equiform with the definition and construct the corresponding thesis as in (2).

(4) C is a defined term introduced into a thesis by directive f or g. Since all constants in the categories of the arguments of C are extensional, the parallel expression can be decided.

Hence the system extended by definition A is extensional.

If B is a propositional constant, then the semantic category of sentences contains no variables, and in fact no variables may appear in any thesis preceding A. After directive f or g has been applied once, every meaningful expression containing at most one propositional constant has been decided. If all meaningful expressions containing at most n propositional constants are decidable (where $n > 0$), then any meaningful expression C which contains $n+$ 1 propositional constants is decidable. For such an expression must be an equivalence each of whose arguments contains at most n propositional constants, and is therefore decidable. Hence C can be proved or disproved by one of the directives a, b, c, or d. Hence all expressions meaningful relative to A are decidable.

If B is not a propositional constant, then no functor exists relative to thesis A which can meaningfully have an argument in the semantic category of B. All meaningful expressions which contain no generalisations or occurrences of B are meaningful relative to the last thesis before A, and hence decidable.

If all meaningful expressions which contain at most n generalisations or occurrences of B are decidable, then any meaningful expression C which contains $n+1$ generalisations or occurrences of B is decidable. For C contains either some generalisation D which is not part of a larger generalisation, or some function E which begins with the term B, which is not part of a generalisation, and which is not the functor of a function.

In the first case, generalisation D corresponds to a finite number of possible substitutions of D with basic constants. Each of these substitutions is decidable, since it contains at most n generalisations or occurrences of B. If all of them are true, D can be proved by directive h, the expression corresponding to C with D replaced by 'V' is decidable, and C can be decided by the extensionality lemma. Otherwise D can be disproved by directive i, the expression corresponding to C with D replaced by 'Λ' is decidable, and C can be decided by the extensionality lemma.

In the second case, function E belongs to a semantic category which is exhausted, since it is either a sentence or an argument in some other category. Moreover, every argument of E belongs to a semantic category which is exhausted. If no argument of E contains an occurrence of B, then each critical expression for E corresponds to a substitution of the *definiens* of A which is meaningful relative to the last thesis preceding A, and so decidable. Hence each critical expression for E is decidable. The explanation of exhaustion assures us that there is a constant having corresponding critical expressions decided in the same way, and this constant is extensionally equivalent to E. The expression corresponding to C but with this constant replacing E therefore contains at most n generalisations or occurrences of B, and is decidable. Therefore C can be decided by the extensionality lemma.

Hence all meaningful expressions are decidable, and the system is complete and extensional relative to A. This means that the system fulfills requirement (1) for completeness.

Any semantic category which exists in the system can be exhausted by defining a finite number of additional constants. System C shows how to exhaust the semantic category of sentences. No functor except the primitive 'ϕ' can appear in any thesis before the semantic category of all of its arguments is exhausted, and the first definition added after the axiom effectively removes this exception. We have also seen in section 4.3 how one may define implication, the term 'ϕ-', in our system of computative protothetic. Using implication and the primitive terms, we can construct a definition of a constant which differs from any existing constant in only one specified critical expression, as described by explanation *E50.* Directive e allows us to add this definition unless there is already a constant whose critical expressions are all decided in the way we wish them to be. Such a definition may, for example have the form

$$\llcorner ab\ldots c\lrcorner\ulcorner\phi\Big(\phi\big(\alpha\,\Phi{<}ab\ldots c{>}\big)\Omega{<}ab\ldots c{>}\Big)\urcorner$$

in which 'Φ' is the existing function, and 'α' is an expression which is true except when the arguments a, b, $\ldots$, and c are extensionally equivalent to the corresponding constant arguments in the specified critical expression. For example, the expression 'α' may have the form

$$\phi\text{-}\Bigg(A_1\phi\text{-}\bigg(A_2\phi\text{-}\Big(\ldots\phi\text{-}\big(A_{n-1}\phi(A_n\Lambda)\big)\ldots\Big)\bigg)\Bigg)$$

in which each antecedent A_x is true only if the corresponding argument of the new constant is extensionally equivalent to the corresponding argument of the specified critical expression. For example, A_n may have the form

$$\llcorner xy\ldots z\lrcorner\ulcorner\phi\big(\phi_n{<}xy\ldots z{>}C_n{<}xy\ldots z{>}\big)\urcorner$$

in which ϕ_n is the nth argument of the defined functor, and C_n is the corresponding argument in the specified critical expression. For example, if it is required to introduce into system C at some point after *C99* a functor '$\dot{o}$' which differs from the primitive term 'ϕ' only in the critical expression '$\phi(\mathrm{VV})$', we could write the definition

$$\llcorner pq\lrcorner\ulcorner\phi\Bigg(\phi\bigg(\phi\text{-}\Big(\phi\text{-}(p\mathrm{V})\phi\big(\phi(q\mathrm{V})\Lambda\big)\Big)\phi(pq)\bigg)\dot{o}(pq)\Bigg)\urcorner$$

The new definition clearly fulfills the requirements, and in a similar fashion any constant required by *E51* can be defined. Therefore every semantic category in the system can be exhausted, and the system fulfills requirement (3) for completeness.

We should be able to define a term in a semantic category which takes any number of arguments in any existing categories and which forms a function belonging to any existing category. To define such a constant, we need only introduce into the system a variable in each of the argument categories, and write a definition. As in the discussion of requirement (3), we define enough constants to exhaust each argument category, and then introduce by directive h a variable into each argument category. Hence the system fulfills all requirements for completeness.

7.4. Equivalence with standard protothetic

We have not gone into great detail about the standard system of protothetic $\mathfrak{S}_5$. Without such detail a full exposition — even a full informal exposition — of the relationship between $\mathfrak{S}_5$ and our system C is not very intelligible. For this reason we shall look in this section only at those parts of the exposition which may shed further light on system C.

In a sense systems C and $\mathfrak{S}_5$ are equivalent, but it is difficult to set up a correspondence between them, since it cannot be a one-to-one correspondence. For example, in $\mathfrak{S}_5$ we can define any number of synonymous constants, but in system C we cannot define a single pair of synonyms. To determine whether one or more constants in standard protothetic correspond to a constant in computative protothetic, we must sometimes investigate the provability of a large number of theses; we cannot give a simple 'structural' correspondence. A full, formal explanation of the correspondence between the two systems is beyond the scope of the present work.

Once a correspondence has been set up, in order to prove the equivalence of the two systems, we must prove that theses corresponding to the axioms of each system are true in the other, and that the directives of each system are valid in the other. Thus, recalling my modification to Leśniewski's metatheorem *L1* discussed in section 3.7, we can prove in system C that

(1) Some axiom system for system $\mathfrak{S}$ is true.

(2) The four laws of implication are true.

(3) The metarule *M5* is valid.

(4) The directives of system $\mathfrak{S}_5$ are valid.

Requirement (1) is satisfied by thesis *C25.* Requirement (2) is satisfied by theses *C96* through *C99.* If the directives of sytem $\mathfrak{S}_5$ are valid, then metarule *M5* can be established using theses *C82* and *C94.* The majority of the burden then falls on proving the validity of the five directives of $\mathfrak{S}_5$:

(α) Since system C is consistent and complete, the detachment directive is valid. If there are two theses of types '$\phi(\alpha\beta)$' and 'α' respectively, then the expression 'β' is meaningful and decidable. If a thesis of the type '$\phi(\beta\Lambda)$' is provable, then by directive b we can prove a thesis of the type '$\phi\big(\phi(\alpha\beta)\Lambda\big)$', and the system is inconsistent. Hence '$\phi(\beta\Lambda)$' is not provable, and 'β' is provable.

(β) To prove that the substitution directive is valid we require directive h and the fact that system C is consistent, complete, and extensional.

(γ) To prove the validity of partitioning the quantifier, we need to use directives a, d, h, and i, together with the proofs of consistency and completeness.

(δ) The definition directive in $\mathfrak{S}_5$ is much more 'liberal' than that of system C. The completeness of the latter system shows that the expressions corresponding to basic substitutions of the *definiens* of an $\mathfrak{S}_5$ definition are decidable in system C, and that there is at least one constant with all of its corresponding critical expressions decided in the same way.

(ϵ) The directive for writing theses of extensionality is valid in system C because of its consistency and completeness, and because of directives a, d, h, and i.

In proving that the directives of system C are valid in $\mathfrak{S}_5$, all directives are fairly simple except h and i. In particular, a thesis corresponding to a definition introduced by directive e is always a valid definition in $\mathfrak{S}_5$, provided we chose an available shape for the new constant. The validity of directive i follows from Tarski's discoveries about quantifiers in standard protothetic[4]. The validity of directive h follows from other results of Tarski[5], but the proof is quite tedious.

Once we have demonstrated that systems C and $\mathfrak{S}_5$ are equivalent, it follows that system $\mathfrak{S}_5$ is consistent and complete. Note that the system $\mathfrak{S}_5$ to which I refer has been modified from that described in Leśniewski29 so that words equiform with the primitive term 'ϕ' are not allowed to be used in my system as a variable.

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[4] Leśniewski29, p. 39–41.

[5] See Tarski56, p. 22, theorem D_2, and Leśniewski29, pp. 41–4.

Bibliography

The bibliography contains frequent references to the following periodicals:

CollLog — *Collectanea Logica*. The first (and only) issue of this periodical was printed in 1939, but all copies were destroyed in the war before they could be distributed. A few offprints of the first five articles survive. SOBOCIŃSKI49 contains an account of the history and contents of this periodical.

NDJFL — *The Notre Dame Journal of Symbolic Logic.*

RPTNO — *Rocznik Polskiego Towarzystwa Naukowego na Obczyźnie.* In some years this periodical was published under the title *Polskie Twarzystwo Naukowe na Obczyźnie: Rocznyk.*

SPTNW — *Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego: Wydział III.* This periodical was also officially named *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie: Classe III.*

Works by joint authors are listed by the name of the first author only. Some articles were published with two titles in different languages; the title in the same language as the article is given below, while the other title is omitted. Some articles appeared in several instalments published in several issues of a periodical or, in one case, in different periodicals. In such cases there is a separate entry for each instalment. When the name of a book or periodical is followed by one or more numbers in parentheses, the first number gives the volume in which the reference appears, the others refer to the issue of the periodical.

AJDUKIEWICZ34 Ajdukiewicz, Kazimierz, 'W sprawie "universaljów"', *Przegląd Filozoficzny* (37–3), 1934, pp. 219–34.

AJDUKIEWICZ35 —, 'Die syntaktische Konnexität', *Studia Philosophica* (1), 1935, pp. 1–27. An English translation of this appears in MCCALL67, pp. 207–31.

AJDUKIEWICZ50 —, 'On the notion of existence', *Studia Philosophica* (4), 1949–50, pp. 7–22.

BIRD75 Bird, Otto Allen, 'Leśniewski, Stanisław', *Encyclopædia Brittanica* 15th. ed. (Macropedia 10), 1975, pp. 832–4.

DEDEKIND88 Dedekind, Richard, *Was sind und was sollen die Zahlen?*, F. Vieweg, 1888.

FREGE79 Frege, Gottlob, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, L. Nebert, 1879.

FREGE93 —, *Grundgesetze der Arithmetik, Begriffsschriftlich abgeleitet* (1), H. Pohle, 1893.

FREGE03 —, — (2), H. Pohle, 1903.

FREGE76 —, *Wissenschaftlicher Briefwechsel*, *ed.* Gottfried Gabriel, Hans Hermes, *et. al.*, Felix Meiner Verlag, 1976.

HENRY72 Henry, D(esmond) P., *Medieval Logic and Metaphysics: a Modern Introduction*, Hutchinson University Library, 1972.

KALMAN78 Kalman, John A., 'A Shortest Single Axiom for the Classical Equivalential Calculus', *NDJFL* (19–1), 1978, pp. 141–4.

KEARNS69 Kearns, John T., 'Two Views of Variables', *NDJFL* (10–2), 1969, pp. 163–80.

KOTARBIŃSKI24 Kotarbiński, Tadeusz, 'Prawdziwość i fałszywość definicyj', *Przegląd Filozoficzny* (27–3,4), 1924, pp. 263–5.

LEBLANC83 Le Blanc, Audoënus V., *A Study of the Axiomatic Foundations of Mereology* (M. A. thesis, University of Manchester), 1983.

LEBLANC85 —, 'Investigations in Protothetic', *NDJFL* (26–4), October, 1985, pp. 483–489.

LEBLANC85A —, 'Axioms for Mereology', *NDJFL* (26–4), October, 1985, pp. 429–436.

LEBLANC85B —, 'New Axioms for Mereology', *NDJFL* (26–4), October, 1985, pp. 437–443.

LEJEWSKI55 Lejewski, Czesław, 'A Contribution to Leśniewski's Mereology', *RPTNO* (5), 1955, pp. 43–50.

LEJEWSKI55A —, 'Logic and Existence', *British Journal for the Philosophy of Science* (5), 1955, pp. 104–19.

LEJEWSKI56 —, review of SŁUPECKI53, *Journal of Symbolic Logic* (21–2), 1956, pp. 188–91.

LEJEWSKI58 —, 'On Leśniewski's Ontology', *Ratio* (1–2), December, 1958, pp. 150–76. A version of this paper appears in SRZEDNICKI84, pp. 123–48.

LEJEWSKI65 —, 'Parts of Speech', *The Aristotelian Society: Supplementary Volume XXXIX*, 1965, pp. 189–204.

LEJEWSKI67 —, 'Leśniewski, Stanisław', *Encyclopedia of Philosophy* (4), 1967, pp. 441–3.

LEJEWSKI68 —, 'A propositional calculus in which three mutually indefinable functors are used as primitive terms', *Studia Logica* (22), 1968, pp. 17–50.

LEJEWSKI69 —, 'Consistency of Leśniewski's Mereology', *The Journal of Symbolic Logic* (34–3), September, 1969, pp. 321–8. This paper was reprinted in SRZEDNICKI84, pp. 231–8.

LEJEWSKI70 —, 'Quantification and Ontological Commitment', *Physics, Logic, and History*, *ed.* W. Yourgrau and A. D. Breck, 1970, pp. 173–81.

LEJEWSKI76 —, 'Ontology and Logic', *Philosophy of Logic*, *ed.* Stephan Körner, 1976, Basil Blackwell, pp. 1–63.

LEJEWSKI77 —, 'Systems of Leśniewski's Ontology with the Functor of Weak Inclusion as the Only Primitive Term', *Studia Logica* (36–4), 1977, pp. 323–349.

LEJEWSKI79 —, 'Idealization of ordinary language for the purposes of logic', in *Function and Context in Linguistic Analysis: A Festschrift for William Haas*, ed. by D. J. Allerton, Edward Carney, and David Holdcroft, Cambridge University Press, 1979, pp. 94–110.

LEJEWSKI81 —, 'Propositional Attitudes and Extensionality', *Philosophie als Wissenschaft: Essays in Scientific Philosophy*, *ed.* Edgar Morscher, Otto Neumaier, and Gerhard Zecher, Comes Verlag, 1981, pp. 211-228.

LEJEWSKI89 —, 'Formalization of Functionally Complete Propositional Calculus with the Functor of Implication as the only Primitive Term', *Studia Logica* (48–4), 1989, pp. 479–94.

Leśniewski16 Leśniewski, Stanisław, *Podstawy ogólnej teoryi mnogości*, part I, volume 2 in the series *Prace Polskiego Koła Naukowego w Moskwie, Sekcya matematyczno-przyrodnicza*, 1916.

Leśniewski27 —, 'O podstawach matematyki', introduction and chapters 1–3, *Przegląd Filozoficzny* (30–2,3), 1927, pp. 164–206. A highly condensed and somewhat inaccurate English translation of this article (including most of its later instalments) appeared in *Topoi* (2–1), June, 1983, pp. 7–52.

Leśniewski28a —, —, chapter 4, *Przegląd Filozoficzny* (31–3), 1928, pp. 261–91.

Leśniewski29a —, —, chapter 5, *Przegląd Filozoficzny* (32–1,2), 1929, pp. 60–101.

Leśniewski30a —, —, chapters 6–9, *Przegląd Filozoficzny* (33–1,2), 1930, pp. 77–105.

Leśniewski31a —, —, chapters 10–11, *Przegląd Filozoficzny* (34–2,3), pp. 142–70.

Leśniewski28 —, 'O podstawach teorji klas. O podstawach ontologji. O podstawach logistyki', *Przegląd Filozoficzny* (31–1,2), 1928, pp. 160–1.

Leśniewski29 —, 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', introduction and §§1–11, *Fundamenta Mathematicae* (14), 1929, pp. 1–81.

Leśniewski39 —, —, §12, *CollLog* (1), 1939, pp. 61–144.

Leśniewski30 —, 'Über die Grundlagen der Ontologie', *SPTNW* (33), 1930, pp. 111–32. By a printer's error, the last four lines of this article appear at the bottom of p. 133.

Leśniewski31 —, 'Über Definitionen in der sogenannten Theorie der Deduktion', *SPTNW* (34), 1931, pp. 289–309. An English translation of this article appears in McCall67, pp. 170–87.

Leśniewski38 —, 'Einleitende Bemerkungen zur Fortsetzung meiner Mitteilung u. d. T. "Grundzüge eines neuen Systems der Grundlagen der Mathematik"', *CollLog* (1), pp. 1–60. The offprints of this article are dated 1938. An English translation of this article appears in McCall67, pp. 116–69.

Łukasiewicz20 Łukasiewicz Jan, 'Logika dwuwartościowa', *Przegląd Filozoficzny* (23), 1920, pp. 189–205. An English translation appears in Łukasiewicz70, pp. 89–109.

Łukasiewicz25 —, 'O pewnym sposobie pojmowania teorji dedukcji', *Przegląd Filozoficzny* (28–1,2), 1925, pp. 134–6.

Łukasiewicz28 —, 'O definicyach w teoryi dedukcyi', *Ruch Filozoficzny* (11), 1928/1929, pp. 177–8.

Łukasiewicz29 —, *Elementy logiki matematycznej*, *ed.* M. Presburger, Komisja Wydawnicza Koła Matematycznego-Fizycznego, 1929. An English translation of part of this work, somewhat edited, appeared as Łukasiewicz63.

Łukasiewicz39 —, 'Der Äquivalenzenkalkül', *CollLog* (1), 1939, pp. 145–69. It has not been possible to obtain a copy of this article. English translations appear in McCall67, pp. 88–115, and in Łukasiewicz70, pp. 250–77.

Łukasiewicz63 —, *Elements of Mathematical Logic*, Pergamon Press, 1963. An edited translation of part of Łukasiewicz29.

Łukasiewicz70 —, *Jan Łukasiewicz: Selected Works*, *ed.* L. Borkowski, North Holland Publishing Company, 1970.

ŁukasiewiczTarski30 Łukasiewicz, J(an), and Tarski, A(lfred), 'Untersuchungen über den Aussagenkalkül', *SPTNW* (33–1,3), 1930, pp. 30–50. English translations of this have appeared in Tarski56, pp. 38–59, and in Łukasiewicz70, pp. 131–52.

Luschei62 Luschei, Eugene, *The Logical Systems of Leśniewski*, North Holland Publishing Company, 1962.

McCall67 McCall, Storrs, *ed.*, *Polish Logic: 1920–1939*, Oxford University Press, 1967.

Meredith51 Meredith, C. A., 'On an Extended System of the Propositional Calculus', *Proceedings of the Royal Irish Academy* (54–A), 1951, pp. 37–47.

Meredith63 Meredith, C. A., and Prior, A. N., 'Notes on the Axiomatics of the Propositional Calculus', *NDJFL* (4-3), 1963, pp. 171–87.

Mihailescu37 Mihailescu, E. Gh., 'Recherches sur un sous-système du calcul des propositions', Annales Scientifiques de l'Université de Jassy (23), 1937, pp. 106–24.

Nicod20 Nicod, J. G. P., 'A Reduction in the number of the Primitive Propositions of Logic', *Proceedings of the Cambridge Philosophical Society* (19), 1920, pp. 32–41.

Peterson76 Peterson, Jeremy George, 'Shortest Single Axioms for the Classical Equivalential Calculus', *NDJFL* (17–2), 1976, pp. 267–71.

Quine40 Quine, W(illard) V(an Orman), review of Leśniewski38 and Leśniewski39, *The Journal of Symbolic Logic* (5–2), June, 1940, pp. 83–84.

Quine52 —, review of Ajdukiewicz50, *Journal of Symbolic Logic* (17–2), 1952, pp. 141–2.

Rickey72 Rickey, V. Frederick, 'Axiomatic Inscriptional Syntax', Part I, *NDJFL* (13–1), January, 1972, pp. 1–33.

Rickey73 —, —, Part II, *NDJFL* (14–1), January, 1973, pp. 1–52.

Rickey77 —, 'A Survey of Leśniewski's Logic', *Studia Logica* (36–4), 1977, pp. 407–26.

Sheffer13 Sheffer, Henry Maurice, 'A set of five independent postulates for Boolean Algebras, with applications to logical constants', *Transactions of the American Mathematical Society* (14–4), October, 1913, pp. 481–8.

Słupecki53 Słupecki, Jerzy, 'St. Leśniewski's Protothetics', *Studia Logica* (1), 1953, pp. 44–111.

Sobociński32 Sobociński, Bolesław, 'Z badań nad teorją dedukcji', *Przegląd Filozoficzny* (35–3,4), 1932, pp. 171–93.

Sobociński34 —, 'O kolejnych uproszczeniach aksjomatyki "Ontologji" Prof. St. Leśniewskiego', *Fragmenty Filozoficzne*, Warsaw, 1934, pp. 144–60. An English translation of this article appeared in McCall67, pp. 188–200.

Sobociński49 —, 'An Investigation of Protothetic', *Cahiers de l'Institut d'Études Polonaises en Belgique* (5), 1949, pp. 1–39.

Sobociński54 —, 'Z badań nad aksjomatyką prototetyki Stanisława Leśniewskiego', *RPTNO* (4), 1954, pp. 18–20.

Sobociński55 —, 'Studies in Leśniewski's Mereology', *RPTNO* (5), 1955, pp. 34–43. Reprinted in Srzednicki84, pp. 217–27.

Sobociński56 —, 'On Well Constructed Axiom Systems', *RPTNO* (6), 1956, pp. 54–65.

Sobociński60 —, 'On the Single Axioms of Protothetic', Introduction and §§ 1–3, *NDJFL* (1–1,2), January, 1960, pp. 52–73.

Sobociński61 —, —, §§4–5, *NDJFL* (2–2), April, 1961, pp. 111–126.

Sobociński61a —, —, §§6–8, *NDJFL* (2–3), July, 1961, pp. 129–148.

Srzednicki84 Srzednicki, Jan, *ed.*, *Leśniewski's Systems: Ontology and Mereology*, Martinus Nijhoff *and* Ossolineum, 1984.

Tarski23 Tarski, Alfred, 'O wyrazie pierwotnym logistyki', *Przegląd Filozoficzny* (26), 1923, pp. 68–89. There are several versions of this paper, which is a summary of Tarski's doctoral thesis. One version appears in Tarski56, pp. 1–23.

Tarski56 —, *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, Clarendon Press, 1956.

Wajsberg37 Wajsberg, Mordchaj, 'Metalogische Beiträge', *Wiadomości Matematyczne* (43), 1937, pp. 1–38. Translated in McCall67, pp. 283–318.

WhiteheadRussell10 Whitehead, Alfred North, and Russell, Bertrand, *Principia Mathematica* (1), Cambridge University Press, 1910. Second edition, 1925.